UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: STK3505/4505 — Problems and methods in Actuarial science

Day of examination: Monday December 10th 2018

Examination hours: 09.00 – 13.00

This problem set consists of 5 pages.

Appendices: None

Permitted aids: Approved calculator

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Introduction

Problem 1 is on general insurance, Problem 2 on life insurance and Problem 3 on financial risk. All of them must be answered and all sub-problems count equally. Write program sketches either in pseudo code or as R commands. The mathematical definitions and expressions below (that you need not prove or justify) may help you solving some of the sub-problems.

Rules of double expectation and variance:

$$\mathrm{E}(Y) = \mathrm{E}(\mathrm{E}(Y|\boldsymbol{X}))$$
$$\mathrm{Var}(Y) = \mathrm{Var}(\mathrm{E}(Y|\boldsymbol{X})) + \mathrm{E}(\mathrm{Var}(Y|\boldsymbol{X})).$$

Mean, standard deviation and cumulative distribution function of the log-logistic distribution: If $Z \sim log-logistic(\alpha, \beta)$, then

$$\mathrm{E}(Z) = \beta\frac{\pi}{\alpha\sin(\pi/\alpha)} \quad \text{and} \quad \mathrm{sd}(Z) = \beta\sqrt{\frac{2\pi}{\alpha\sin(2\pi/\alpha)} - \left(\frac{\pi}{\alpha\sin(\pi/\alpha)}\right)^2}$$

and

$$F(z) = \frac{1}{1+\left(\frac{z}{\beta}\right)^{-\alpha}}.$$

Black-Scholes formula for a put option over $(0, T)$ with guarantee r_g and initial value of the stock $v_0 = 1$:

$$\pi(1) = (1+r_g)e^{-rT}\Phi(a) - \Phi(a - \sigma\sqrt{T})$$

where $a = \frac{\log(1+r_g) - rT + \sigma^2 T/2}{\sigma\sqrt{T}}$. Here the volatility and the discount over $(0, T)$ are $\sigma\sqrt{T}$ and e^{-rT}, respectively.

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Problem 1 General insurance

Let $\mathcal{X} = Z_1 + \ldots + Z_{\mathcal{N}}$ be the total pay-out in a general insurance portfolio under standard assumptions, where $\mathcal{N} \sim Poisson(\lambda)$ is independent of the individual losses Z_i, which are independent, identically $log - logistic(\alpha, \beta)$ distributed.

a

Derive a sampler for the log-logistic distribution using the inversion method.

b

Sketch a program for generating m samples of $\mathcal{X}$. Explain how to determine $E(\mathcal{X})$ and $sd(\mathcal{X})$, as well as the reserve from these simulations (remember to define what the reserve is).

The table below shows the expectation, standard deviation and a few percentiles of the distribution of $\mathcal{X}$ when $\lambda = 10$, $\alpha = 3$, $\beta = 0.83$. The number of simulations was $100,000$:

$E(\mathcal{X})$	$sd(\mathcal{X})$	1%	5%	25%	50%	75%	95%	99%
10.0	4.0	2.7	4.4	7.2	9.6	12.3	17.2	21.6

c

Derive expressions for $E(\mathcal{X})$ and $sd(\mathcal{X})$ for the given parameters (**Hint**: use the rules of double expectation and variance). Compare the exact values you get for $E(\mathcal{X})$ and $sd(\mathcal{X})$ with the ones from the table. Are there any signs of errors in the simulation program used for computing the values in the table?

d

What are the 95% and 99% reserves of the portfolio?

Suppose the portfolio is reinsured through a contract where the reinsurer responsibility per event is

$$Z^{re} = \begin{cases} 0, & Z \leq a \\ \theta(Z - a), & Z > a \end{cases},$$

where $a, \theta > 0$ are fixed by the contract.

e

Modify the program in **b** so that you can determine cedent net reserve with the reinsurance reimbursement taken into account.

The cedent net responsibility $\mathcal{X}^{ce}$ is summarized in the table below for $a = 1$, $\theta = 0.6$ and other parameters as before.

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$E(\mathcal{X}^{ce})$	$sd(\mathcal{X}^{ce})$	1%	5%	25%	50%	75%	95%	99%
7.7	2.6	2.4	3.7	5.9	7.5	9.3	12.1	14.3

f

What do the 95% and 99% cedent net reserves become with reinsurance? Comment on the difference from **d**. What is the **reinsurance** pure premium and what is the actual reinsurance premium when the loading is $\gamma = 1$?

g

Assume that the claim numbers $N_1, \ldots, N_J$ from the J policies constituting the portfolio are independent and follow a Poisson distribution with parameter μT, with μ fixed and T the time period, such that the parameter λ of the distribution of $\mathcal{N} = N_1 + \ldots + N_J$ is $J\mu T$. Use the results from **c** to show that $\frac{\mathrm{sd}(\mathcal{X})}{\mathrm{E}(\mathcal{X})} \underset{J\to\infty}{\longrightarrow} 0$.

h

What if μ instead is random with $0 < \mathrm{E}(\mu), \mathrm{sd}(\mu) < \infty$, and $N_1, \ldots, N_J|\mu \sim Poisson(\mu T)$; what is $\underset{J\to\infty}{lim} \frac{\mathrm{sd}(\mathcal{X})}{\mathrm{E}(\mathcal{X})}$ now? Comment on the difference from **g**.

Problem 2 Life insurance

Suppose a pension starts to run at age l_r, lasting until the individual dies, with an amount s paid out at the beginning of each period. The probability of an individual of age l living at least k periods longer is ${}_kp_l$, and the discount per period is d.

a

Write down an expression for the expected present value π_{l_0} of such a pension for an individual at age l_0 when $l_0 < l_r$ and when $l_0 \geq l_r$, where you explain the different factors of the expression and how the expression is obtained.

The table below shows π_{l_0} when $s = 1$, $l_0 = 30$ and the age of retirement is varied between $l_r = 62, 67, 72$,

$$\log({}_kp_l) = -\theta_0 k - \frac{\theta_1}{\theta_2}\left(e^{\theta_2 k} - 1\right)e^{\theta_2 l},$$

with $\theta_0 = 0.009$, $\theta_1 = 0.000046$ and $\theta_2 = 0.0908$ and the discount is $d = 1/(1+r)$, with $r = 0.03$. The order of the corresponding π_{l_0} has been changed.

l_r	??	??	??
π_{l_0}	2.23	3.38	1.37

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b

Identify which values of l_r correspond to those of π_{l_0} and justify your argument.

c

Assume that the pension is financed by fixed contributions ζ at the start of each year from age $l_0 < l_r$ up to age $l_r - 1$. What is the expected present value of all these payments? Explain the different factors of the expression.

d

What does it mean that ζ is determined by equivalence (explain in words)? Write down a mathematical expression for it when the contributions are made at the start of each period up to one period before retirement.

Problem 3 Financial risk

Assume that one has invested in an equity portfolio consisting of 4 stocks with corresponding returns

$$R_j = e^{\xi_j + \sigma_j \varepsilon_j} - 1, \quad \varepsilon_j \sim N(0,1),$$

where $\mathrm{Cor}(\varepsilon_i, \varepsilon_j) = \rho_{ij}$, $i, j = 1, \ldots, 4$, and weights $w_1, \ldots, w_4$.

The table below shows the mean, standard deviation and some quantiles of the portfolio return $\mathcal{R}$

Scenario	$E(\mathcal{R})$	$sd(\mathcal{R})$	1%	5%	25%	50%	75%	95%	99%
??	0.15	0.18	-0.21	-0.12	0.02	0.14	0.27	0.48	0.64
1	0.15	0.29	-0.37	-0.26	-0.06	0.12	0.32	0.69	1.00
??	0.15	0.27	-0.34	-0.23	-0.04	0.12	0.31	0.64	0.93
??	0.15	0.15	-0.15	-0.07	0.05	0.14	0.25	0.41	0.53

for the following 4 scenarios:

Scenario	Correlations	Weights
1	$\rho_{ij} = 0, i \neq j$	$1, 0, 0, 0$
2	$\rho_{ij} = 0, i \neq j$	$0.25, 0.25, 0.25, 0.25$
3	$\rho_{ij} = 0.2, i \neq j$	$0.25, 0.25, 0.25, 0.25$
4	$\rho_{ij} = 0.8, i \neq j$	$0.25, 0.25, 0.25, 0.25$

when the remaining parameters are $\xi_j = 0.11$ and $\sigma_j = 0.25$, $j = 1, \ldots, 4$.

a

Identify which scenario corresponds to which row in the figure and justify your answers. Comment on the difference between the 4 investments.

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b

Assume that the actual investment is Scenario 3, and that one wishes to protect this investment with a put option on the portfolio return with guaranteed return r_g. Sketch a program for simulating the pay-off $X = \max(r_g - \mathcal{R}, 0)$, where you do not need to go into details on how to simulate from a multivariate normal distribution. Explain how you could use that to compute the risk-neutral price $\pi(v_0)$ when the discount is e^{-r} during the period $T = 1$ (remember that in the risk-neutral model, $\xi_j = r - \frac{1}{2}\sigma_j^2$).

c

Can you think of an alternative method for computing $\pi(v_0)$ if the actual investment is Scenario 1?

END