Solutions to the exam in STK2520
Autumn 2013

Problem 1 (i) We know that the inter-arrival times W_i are $i.i.d$ with common distribution $Exp(\lambda)$. Then maximization of the likelihood function yields the MLE

$$\widehat{\lambda} = \frac{n}{\sum_{i=1}^{n} W_i}$$

of the jump intensity of $N(t)$.

So we get for $n = 8$ and $T_8 = \sum_{i=1}^{8} W_i = 17$ days (06/30/1990 is excluded, i.e. $W_1 = 1, W_2 = 4, ..., W_8 = 1$)

$$\widehat{\lambda} = \frac{8}{17}.$$

(ii)

$$P(W_1 > 4) = \exp(-4\widehat{\lambda}) \approx 0.152$$

$$P(T_2 > 3) = \exp(-3\widehat{\lambda})(1 + \frac{3\widehat{\lambda}}{1}) \approx 0.588,$$

since $T_n \sim \Gamma(n, \lambda)$.

(iii) Using the sample mean we obtain for $t = 365$ days that

$$E[X_1] \approx 4.27836 \text{ mio DKK.}$$

So

$$E[S(t)] = \frac{8}{17} \cdot 365 \cdot 4.27836 = 734.87125 \text{ mio DKK.}$$

Hence

$$p_{EV}(t) = 1.27 \cdot 734.871 = 933.28648 \text{ in mio DKK.}$$

Problem 2

(i) Because of the equivalence principle Π_0 satisfies the equation

$$\sum_{k=0}^{3} c_{k+1} v^{k+1} \cdot_k p_x q_{x+k} + E v^4 \cdot_4 p_x - \sum_{k=0}^{3} \Pi_k v^k \cdot_k p_x = 0,$$

where $c_{k+1} = k + 1, k = 0, 1, 2, 3$. Solving the latter equation gives

$$\Pi_0 = 0.559043.$$

(ii) Apply the discrete version of Thiele's differential equation, that is

$$ {}_kV + \Pi_k = v\left[c_{k+1} \cdot q_{x+k} + \ {}_{k+1}V \cdot p_{x+k}\right] $$

to obtain ${}_1V = 0.568039, {}_2V = 1.17399, {}_3V = 1.81778, {}_4V = 2.5 = E$.

Problem 3 By assumption $S_i(t), i = 1, 2$ are independent compound Poisson r.v.'s (for $t = 1$). Then we know that the r.v.

$$ \widetilde{S}(t) = S_1(t) + S_2(t) $$

is compound Poisson with

$$ \widetilde{S}(t) \stackrel{d}{=} \sum_{j=1}^{N_\lambda} Y_j. $$

Here $N_\lambda \sim Pois(\lambda), \lambda = \lambda_1 + \lambda_2 = \frac{16}{21}$ and the $i.i.d$ r.v.'s Y_j are given by the "mixture distribution", that is

$$ P(Y_1 = x) = p_1 P(X_1^{(1)} = x) + p_2 P(X_1^{(2)} = x), $$

where

$$ p_i = \frac{\lambda_i}{\lambda_1 + \lambda_2}, \ i = 1, 2. $$

See the book of Thomas Mikosch.

The latter yields

$$ P(Y_1 = x) = \begin{cases} 0 & , x \neq 1, 2, 3 \\ 3/8 & x = 1 \\ 5/16 & x = 2 \\ 5/16 & x = 3 \end{cases} $$

We want to employ the Panjer recursion scheme in the case of Poisson distributed claim numbers:

$$ p_n = \sum_{i=1}^{n} \frac{16i}{21n} \cdot P(Y_1 = i) \cdot p_{n-i}, \ n \geq 1, $$

with

$$ p_0 = P(N_\lambda = 0) = e^{-\lambda} \approx 0.466777. $$

Hence

$$ \begin{aligned} P(\widetilde{S}(1) &= 1) \approx 0.1334, P(\widetilde{S}(1) = 2) \approx 0.1302, P(\widetilde{S}(1) = 3) \approx 0.1447 \\ (, ..., P(\widetilde{S}(1) &= 10) \approx 0.0009), \end{aligned} $$

that is

$$ \begin{aligned} P(\widetilde{S}(1) &\leq 1) \approx 0.6001, P(\widetilde{S}(1) \leq 2) \approx 0.7303, P(\widetilde{S}(1) \leq 3) \approx 0.8750 \\ (,...,P(\widetilde{S}(1) &\leq 10) \approx 0.9993) \end{aligned} $$

Problem 4

(i) We have

$$\begin{aligned} {}_{20}p_x &= \exp(-\int_0^{20} (\frac{1}{85-t} + \frac{3}{105-t}) dt) \\ &= \frac{65}{85}(\frac{85}{105})^3 \approx 0.406 \end{aligned}$$

(ii)

$$E[T] = \frac{\alpha}{\lambda} = 15 \text{ (år)}$$

(iii) Since

$$\begin{aligned} \Pr(S &< t) = \sum_{l\geq 0}(G(l+t) - G(l)) \\ &= \sum_{l\geq 0}(e^{-\lambda l}\sum_{k=0}^{2}\frac{(\lambda l)^k}{k!} - e^{-\lambda(l+t)}\sum_{k=0}^{2}\frac{(\lambda(l+t))^k}{k!}) \\ &= \sum_{l\geq 0} e^{-\lambda l} + \lambda \sum_{l\geq 0} e^{-\lambda l} l + \frac{\lambda^2}{2}\sum_{l\geq 0} e^{-\lambda l} l^2 - e^{-\lambda t}\sum_{l\geq 0} e^{-\lambda l} \\ &\quad -e^{-\lambda t}\lambda \sum_{l\geq 0} e^{-\lambda l} l - e^{-\lambda t}\lambda t \sum_{l\geq 0} e^{-\lambda l} \\ &\quad -e^{-\lambda t}\frac{\lambda^2}{2}\sum_{l\geq 0} e^{-\lambda l} l^2 - e^{-\lambda t}\frac{\lambda^2}{2} 2t \sum_{l\geq 0} e^{-\lambda l} l \\ &\quad -e^{-\lambda t}\frac{\lambda^2}{2} t^2 \sum_{l\geq 0} e^{-\lambda l}. \end{aligned}$$

We know that

$$\begin{aligned} \sum_{l\geq 0} q^l &= \frac{1}{1-q}, \\ \sum_{l\geq 1} q^{l-1}\cdot l &= \frac{d}{dq}\left(\frac{1}{1-q}\right) \\ \sum_{l\geq 2} q^{l-2}\cdot l \cdot (l-1) &= \frac{d^2}{dq^2}\left(\frac{1}{1-q}\right), 0 \leq q < 1. \end{aligned}$$

So

$$\sum_{l\geq 0} q^l \cdot l = \sum_{l\geq 1} q^l \cdot l = q \cdot \frac{d}{dq}\left(\frac{1}{1-q}\right) = \frac{q}{(1-q)^2}$$

and

$$\sum_{l\geq 0} q^l \cdot l^2 = \sum_{l\geq 1} q^l \cdot l + \sum_{l\geq 2} q^l \cdot l \cdot (l-1)$$
$$= \frac{q}{(1-q)^2} + \frac{2q^2}{(1-q)^3}.$$

Hence we obtain for $q = e^{-\lambda}$ that

$$\begin{aligned}
\Pr(S &< t) = \sum_{l\geq 0}(G(l+t) - G(l)) \\
&= \sum_{l\geq 0}(e^{-\lambda l}\sum_{k=0}^{2}\frac{(\lambda l)^k}{k!} - e^{-\lambda(l+t)}\sum_{k=0}^{2}\frac{(\lambda(l+t))^k}{k!}) \\
&= \frac{1}{1-e^{-\lambda}} + \lambda(\frac{e^{-\lambda}}{(1-e^{-\lambda})^2}) + \frac{\lambda^2}{2}(\frac{e^{-\lambda}}{(1-e^{-\lambda})^2} + \frac{2e^{-2\lambda}}{(1-e^{-\lambda})^3}) - e^{-\lambda t}(\frac{1}{1-e^{-\lambda}}) \\
&\quad -e^{-\lambda t}\lambda(\frac{e^{-\lambda}}{(1-e^{-\lambda})^2}) - e^{-\lambda t}\lambda t(\frac{1}{1-e^{-\lambda}}) \\
&\quad -e^{-\lambda t}\frac{\lambda^2}{2}(\frac{e^{-\lambda}}{(1-e^{-\lambda})^2} + \frac{2e^{-2\lambda}}{(1-e^{-\lambda})^3}) - e^{-\lambda t}\frac{\lambda^2}{2}2t(\frac{e^{-\lambda}}{(1-e^{-\lambda})^2}) \\
&\quad -e^{-\lambda t}\frac{\lambda^2}{2}t^2(\frac{1}{1-e^{-\lambda}}).
\end{aligned}$$

Then integration by parts gives

$$\begin{aligned}
&E[S] \\
&= \int_0^1 \Pr(S > t)dt \\
&= (1 - \{\frac{1}{1-e^{-\lambda}} + \lambda(\frac{e^{-\lambda}}{(1-e^{-\lambda})^2}) + \frac{\lambda^2}{2}(\frac{e^{-\lambda}}{(1-e^{-\lambda})^2} + \frac{2e^{-2\lambda}}{(1-e^{-\lambda})^3}) \\
&\quad -\frac{1-e^{-\lambda}}{\lambda}(\frac{1}{1-e^{-\lambda}} + \lambda(\frac{e^{-\lambda}}{(1-e^{-\lambda})^2}) + \frac{\lambda^2}{2}(\frac{e^{-\lambda}}{(1-e^{-\lambda})^2} + \frac{2e^{-2\lambda}}{(1-e^{-\lambda})^3})) \\
&\quad -\frac{1}{\lambda^2}(1-e^{-\lambda}(1+\lambda))(\lambda(\frac{1}{1-e^{-\lambda}}) + \frac{\lambda^2}{2}2(\frac{e^{-\lambda}}{(1-e^{-\lambda})^2})) \\
&\quad -\frac{2}{\lambda^3}(1-e^{-\lambda}(1+\lambda+\frac{\lambda^2}{2}))\frac{\lambda^2}{2}(\frac{1}{1-e^{-\lambda}})\}) \\
&= 0.500051.
\end{aligned}$$

Problem 5 (i) Claim:

$$(X_{(1)}, ..., X_{(n)}) \stackrel{d}{=} (\frac{X_n}{n}, \frac{X_n}{n} + \frac{X_{n-1}}{n-1}, ..., \frac{X_n}{n} + \frac{X_{n-1}}{n-1} + ... + \frac{X_2}{2}, \frac{X_n}{n} + \frac{X_{n-1}}{n-1} + ... + \frac{X_1}{1}). \quad (1)$$

Without loss of generality let us verify the latter relation for $n = 2$: Define $L(x_1, x_2) = (\frac{x_2}{2}, \frac{x_2}{2} + \frac{x_1}{1}) = (\frac{x_2}{2}, \frac{x_2}{2} + x_1)$. Then one observes that the function L has an inverse L^{-1} given by $L^{-1}(y_1, y_2) = (y_2 - y_1, 2y_1)$. So we get

$$\begin{aligned}
P(\frac{X_2}{2} &\leq x_1, \frac{X_2}{2} + \frac{X_1}{1} \leq x_2) = P(L(X_1, X_2) \in [0, x_1] \times [0, x_2]) = P((X_1, X_2) \in L^{-1}([0, x_1] \times [0, x_2])) \\
&= \int_{L^{-1}([0,x_1]\times[0,x_2])} f_{X_1,X_2}(u_1, u_2) du_1 du_2 \\
&\overset{\text{subst.:}(u_1,u_2)=L^{-1}(y_1,y_2)}{=} \int_0^{x_1} \int_0^{x_2} 2 \cdot f_{X_1,X_2}(L^{-1}(y_1, y_2)) dy_1 dy_2 \\
&\overset{X_1,X_2 \text{ indep.}}{=} \int_0^{x_1} \int_0^{x_2} 2\lambda e^{-\lambda(y_2-y_1)} \lambda e^{-\lambda(2y_1)} 1_{\{y_2>y_1\}} dy_1 dy_2 \\
&= \int_0^{x_1} \int_0^{x_2} 2! f_{X_1}(y_1) f_{X_2}(y_2) 1_{\{y_2>y_1\}} dy_1 dy_2 \\
&\overset{\text{formulary: k)}}{=} \int_0^{x_1} \int_0^{x_2} f_{X_{(1)},X_{(2)}}(y_1, y_2) dy_1 dy_2.
\end{aligned}$$

Hence the claim is valid for $n = 2$.

Define

$$G(x_1, ..., x_n) = \sum_{i=1}^{k-1} x_{n-i+1} - (k-1) \cdot x_{n-k+1}.$$

Then it follows from (1) for all $n \geq k \geq 2$ that

$$\begin{aligned}
\sum_{i=1}^{n} (X_{(n-i+1)} - X_{(n-k+1)})_+ &= \sum_{i=1}^{k-1} X_{(n-i+1)} - (k-1) \cdot X_{(n-k+1)} = G(X_{(1)}, ..., X_{(n)}) \\
\overset{d}{=} G(\frac{X_n}{n}, ..., \frac{X_n}{n} + \frac{X_{n-1}}{n-1} + ... + \frac{X_1}{1}) &= X_1 + ... + X_{k-1} \sim \Gamma(k-1, \lambda) \text{ (see Prob. 1).}
\end{aligned}$$

Hence

$$P(R(t) \leq x \,|\, N(t) \geq k) = P(X_1 + ... + X_{k-1} \leq x).$$

(ii) So

$$P(R(t) \leq 5 \,|\, N(t) \geq 5) = 1 - e^{-5} \sum_{j=0}^{3} \frac{(5)^j}{j!} \approx 0.735.$$