UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in:	STK4011/STK9011 - Statistical Inference Theory —
Day of examination:	17.12.25
Examination hours:	9.00 – 13.00
This problem set consists of 3 pages.	
Appendices:	None.
Permitted aids:	Approved calculator and "List of formulas for STK4011/9011 Statistical Inference Theory (2025)"

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

Assume Y being a Beta(α, β) distributed random variable, with probability density function

$$f(y;\alpha,\beta) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} y^{\alpha-1}(1-y)^{\beta-1} & \text{if } 0 < y < 1 \\ 0 & \text{otherwise} \end{cases},$$

where $\alpha > 0$ and $\beta > 0$. The moment generating function is

$$M_Y(t) = 1 + \sum_{k=1}^{\infty} \left(\prod_{r=0}^{k-1} \frac{\alpha + r}{\alpha + \beta + r} \right) \frac{t^k}{k!}.$$

a

Derive a moment estimator for α and β (you may need to remember that $0^0 = 1$).

b

Show that $\log y$ and $\log(1-y)$ are sufficient statistics for α and β, respectively. What would be the sufficient statistic for the parameter of the Beta distribution in the case $\alpha = \beta$?

c

Show that $\lim_{n\to\infty} n\text{Beta}(1, n) = \text{Exponential}(1)$. To do this, you may want to first show that, when $X_1, \ldots, X_n$ are independent and identically distributed Beta$(1, 1)$ random variables, $X_{(1)} \sim \text{Beta}(1, n)$.
It may be useful to remember that $\lim_{n\to\infty}(1 - a/n)^n = e^{-a}$, and that the cumulative distribution function of an exponential random variable Z is

$$F_Z(z) = 1 - e^{-z}.$$

(Continued on page 2.)

d

The distribution of Y defined in the exercise only allows the random variable to assume values in $(0, 1)$. To allow values in a general interval (a, b), we can consider two additional parameters, a and b, and use the linear transformation $x = y(b - a) + a$. Find the probability density function of $X \sim \text{Beta}(\alpha, \beta, a, b)$.

Problem 2

Assume $Y_1, \ldots, Y_n$ be independent and identically distributed negative binomial random variables, with density mass function

$$f(y; r, p) = \binom{y + r - 1}{y} (1 - p)^y p^r,$$

where r is the number of successes, y is the number of failures, and p is the probability of success on each trial. For this exercise, consider r as known.

a

Show that the moment generating function for the negative binomial random variable is equal to

$$M_Y(t) = \left(\frac{p}{1 - (1 - p)e^t} \right)^r, \text{ for } t < -\log(1 - p).$$

It may be useful to remember the formula of the generalized binomial series, namely $(1 - a)^{-b} = \sum_{k=0}^{\infty} \binom{b+k-1}{k} a^k$.

b

Show that $Z = \sum_{i=1}^{n} Y_i$ is a complete and sufficient statistics for p, and show that it is also distributed as a negative binomial.

c

Compute the maximum likelihood estimator for p and its asymptotic variance.

d

It can be shown that the maximum likelihood estimator for p is biased. You do NOT need to do it, but, instead, show that the maximum likelihood estimator of $1/p$ is unbiased. What is the lowest possible variance for an unbiased estimator of $1/p$?

Problem 3

Let $Y_1, \ldots, Y_n$ be independent and identically distributed Bernoulli variables with success probability p, i.e.,

$$f(y; p) = p^y (1 - p)^{1-y}.$$

(Continued on page 3.)

a

Show that $T = \sum_{i=1}^{n} Y_i$ is a sufficient statistics for p and that the family of pmfs of T has the monotone likelihood ratio property.

b

Find the uniform most powerful test of $H_0 : p \leq \pi$ versus $H_1 : p > \pi$, and describe how you may determine the size of the test.

c

Derive the likelihood ratio test of $H_0 : p = \pi$ versus $H_1 : p \neq \pi$ and describe how you can find an approximate $1 - \alpha$ confidence interval for p based on it.

THE END