

Exercise 1.49

(a) X, Y indep r.v., th: distr of X and $X+Y$

$$Z = X+Y \Rightarrow M_Z(t) = M_X(t)M_Y(t) \Rightarrow M_Y(t) = \frac{M_Z(t)}{M_X(t)}$$

$$X \sim \mathcal{N}(0, 1), \quad X+Y \sim \mathcal{N}(0, 2)$$

$$M_Y(t) = \frac{e^{\frac{2t^2}{2}}}{e^{\frac{t^2}{2}}} = e^{\frac{2t^2 - t^2}{2}} = e^{\frac{t^2}{2}} \Rightarrow Y \sim \mathcal{N}(0, 1)$$

(b) $X \sim \text{Poisson}(\lambda)$, $X+Y \sim \text{Poisson}(\theta)$

$$M_Y(t) = \frac{e^{\theta(e^t - 1)}}{e^{\lambda(e^t - 1)}} = e^{(\theta - \lambda)(e^t - 1)} \Rightarrow Y \sim \text{Poisson}(\theta - \lambda)$$

(c) $X \sim \chi_a^2$, $Z = X+Y \sim \chi_{a+b}^2$

$$M_Y(t) = \frac{(1-2t)^{-a/2}}{(1-2t)^{-(a+b)/2}} = (1-2t)^{-b/2} \Rightarrow Y \sim \chi_b^2$$

(d) $X_1, \dots, X_n$ iid $\mathcal{N}(0, 1)$ $\bar{X} = \frac{\sum X_i}{n}$, s.t. $\sqrt{n}\bar{X} \sim \mathcal{N}(0, 1)$

• Show that $X_i - \bar{X}$ is indep of $\bar{X}$

• Show from $\sum_{i=1}^n X_i^2 = Z_0 + n\bar{X}^2$, $Z_0 = \sum (X_i - \bar{X})^2$ that $Z_0 \sim \chi_{n-1}^2$

• $X_i - \bar{X} \sim \mathcal{N}(0, \sigma^2)$ all linear combinations of Gaussians,
 $\bar{X} \sim \mathcal{N}(0, \frac{1}{n})$ i.e., $(X_i - \bar{X}, \bar{X}) \sim \mathcal{N}_2(0, \Sigma)$

$$\text{Cov}(X_i - \bar{X}, \bar{X}) = E[(X_i - \bar{X})\bar{X}] - E[X_i - \bar{X}]E[\bar{X}]$$

$$= E\left[X_i \frac{1}{n} \sum_{j=1}^n X_j\right] - E[\bar{X}^2]$$

$$= \frac{E[X_i^2]}{n} + \frac{(n-1)}{n} E[X_i X_{\neq i}] - \frac{1}{n} = 0$$

because the properties of Gaussians, $\text{Cov} \neq 0 \Rightarrow$ indep.

$$\bullet \sum_{i=1}^n X_i^2 = Z_0 + n\bar{X}^2 \Rightarrow Z_0 = \sum_{i=1}^n X_i^2 - (n\bar{X})^2 = \chi_n^2 - \chi_1^2 = \chi_{n-1}^2$$

Exercise 1.57

(a) $N(\mu, \sigma^2)$ $f_Y(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}(y-\mu)^2\right\}$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma^2} \exp\left\{-\frac{\mu^2}{2\sigma^2}\right\} \exp\left\{\frac{\mu}{\sigma^2}y - \frac{1}{2\sigma^2}y^2\right\}$$

$\underbrace{\frac{1}{\sqrt{2\pi}}}_{w_1(\mu, \sigma^2)} \underbrace{\frac{1}{\sigma^2}}_{t_1(x)} \exp\left\{-\frac{\mu^2}{2\sigma^2}\right\}_{w_2(\mu, \sigma^2)} \exp\left\{\frac{\mu}{\sigma^2}y - \frac{1}{2\sigma^2}y^2\right\}_{t_2(x)}$

(b) $(X, Y) \sim N_2\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}\right)$

$$f_{X,Y}(x,y) = \frac{1}{2\pi |\Sigma|^{1/2}} \exp\left\{-\frac{1}{2} \begin{pmatrix} x-\mu_1 \\ y-\mu_2 \end{pmatrix}^T \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}^{-1} \begin{pmatrix} x-\mu_1 \\ y-\mu_2 \end{pmatrix}\right\}$$

$$= \frac{1}{2\pi\sigma_1\sigma_2(1-\rho^2)^{1/2}} \exp\left\{-\frac{1}{2(1-\rho^2)} \left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right) + \left(\frac{y-\mu_2}{\sigma_2}\right)^2 \right]\right\}$$

$$= \frac{1}{2\pi\sigma_1\sigma_2(1-\rho^2)^{1/2}} \exp\left\{-\frac{1}{2(1-\rho^2)} \left[\frac{x^2}{\sigma_1^2} - \frac{2x\mu_1}{\sigma_1^2} + \frac{\mu_1^2}{\sigma_1^2} - 2\rho\left(\frac{x}{\sigma_1\sigma_2} - \frac{x\mu_2}{\sigma_1\sigma_2} - \frac{y\mu_1}{\sigma_1\sigma_2} + \frac{\mu_1\mu_2}{\sigma_1\sigma_2}\right) + \frac{y^2}{\sigma_2^2} - \frac{2y\mu_2}{\sigma_2^2} + \frac{\mu_2^2}{\sigma_2^2} \right]\right\}$$

$$= \frac{\exp\left\{-\frac{(\mu_1^2/\sigma_1^2 - 2\mu_1\mu_2/\sigma_1\sigma_2 + \mu_2^2/\sigma_2^2)}{2(1-\rho^2)}\right\}}{2\pi\sigma_1\sigma_2(1-\rho^2)^{1/2}} \exp\left\{\frac{\sigma_2^2\mu_1 + \rho\sigma_1\mu_2}{\sigma_2^2\sigma_1(1-\rho^2)}x - \frac{1}{2(1-\rho^2)\sigma_1^2}x^2\right.$$

$$\left. + \frac{\rho}{(1-\rho^2)\sigma_1\sigma_2}xy + \frac{\sigma_1\mu_2 - \rho\sigma_2\mu_1}{\sigma_1\sigma_2^2(1-\rho^2)}y - \frac{1}{2(1-\rho^2)\sigma_2^2}y^2\right\}$$

$$\eta_1 = \frac{\mu_1/\sigma_1^2 - 2\rho\mu_2/\sigma_1\sigma_2}{2(1-\rho^2)}$$

$$t_1 = x$$

$$\eta_2 = \frac{\mu_2/\sigma_2^2 - 2\rho\mu_1/\sigma_1\sigma_2}{2(1-\rho^2)}$$

$$t_2 = y$$

$$\eta_3 = -\frac{1}{2(1-\rho^2)\sigma_1^2}$$

$$t_3 = x^2$$

$$\eta_4 = -\frac{1}{2(1-\rho^2)\sigma_2^2}$$

$$t_4 = y^2$$

$$\eta_5 = \frac{\rho}{(1-\rho^2)\sigma_1\sigma_2}$$

$$t_5 = xy$$

$$(c) \quad \frac{\mu_1}{(1-\rho^2)\sigma_1^2} = \alpha$$

$$\mu_1 = (1-\rho^2)\sigma_1^2 \alpha$$

$$\frac{1}{(1-\rho^2)\sigma_1^2} = \beta$$

$$\sigma_1^2 = \frac{1}{(1-\rho^2)\beta}$$

$$\frac{\mu_2}{(1-\rho^2)\sigma_2^2} = \gamma$$

$$\mu_2 = (1-\rho^2)\sigma_2^2 \gamma$$

$$\frac{1}{(1-\rho^2)\sigma_2^2} = \delta$$

$$\sigma_2^2 = \frac{1}{(1-\rho^2)\delta}$$

$$\frac{\rho}{(1-\rho^2)\sigma_1\sigma_2} = \varepsilon$$

$$\frac{\rho}{(1-\rho^2)} = \varepsilon \sigma_1 \sigma_2$$

$$\frac{\rho}{(1-\rho^2)} = \frac{\varepsilon}{(1-\rho^2) \beta^{1/2} \delta^{1/2}} = \frac{\varepsilon}{\beta^{1/2} \delta^{1/2}}$$

$$\mu_1 = \frac{(1-\rho^2)\alpha}{(1-\rho^2)\beta} = \frac{\alpha}{\beta}$$

$$\mu_2 = \frac{(1-\rho^2)\gamma}{(1-\rho^2)\delta} = \frac{\gamma}{\delta}$$

$$\sigma_1^2 = \frac{1}{(1 - \frac{\varepsilon}{\delta\beta})\beta}$$

$$\sigma_2^2 = \frac{1}{(1 - \frac{\varepsilon}{\delta\beta})\delta}$$

$$(d) \frac{1}{2\pi\sigma_1\sigma_2(1-\rho^2)^{1/2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{x^2}{\sigma_1^2} - \frac{2x\mu}{\sigma_1^2} + \frac{\mu^2}{\sigma_1^2} + \frac{y^2}{\sigma_2^2} - \frac{2y\mu}{\sigma_2^2} + \frac{\mu^2}{\sigma_2^2} - 2\rho\left(\frac{xy}{\sigma_1\sigma_2} - \frac{x\mu}{\sigma_1\sigma_2} - \frac{y\mu}{\sigma_1\sigma_2} + \frac{\mu^2}{\sigma_2^2}\right)\right]\right\}$$

$$= \frac{1}{2\pi} \frac{\exp\left\{-\frac{1}{2(1-\rho^2)}\left(\frac{\mu^2(1-\rho)}{\sigma^2}\right)\right\}}{\sigma^2(1-\rho^2)^{1/2}} \cdot \exp\left\{-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma^2} + \frac{y^2}{\sigma^2} + 2\frac{x\mu}{\sigma^2} + 2\frac{y\mu}{\sigma^2} - 2\rho\frac{xy}{\sigma^2} - 2\rho\frac{x\mu}{\sigma^2} - 2\rho\frac{y\mu}{\sigma^2}\right)\right\}$$

$$\exp\left\{-\frac{1}{2\sigma^2(1-\rho^2)}\left(x^2 + y^2 + 2x\mu + 2y\mu - 2\rho xy - 2\rho\mu x - 2\rho\mu y\right)\right\}$$

$$= \frac{1}{2\pi} c(\mu, \sigma^2, \rho) \exp\left\{-\frac{1}{2\sigma^2(1-\rho^2)}(x^2 + y^2) + \frac{\mu}{(1-\rho)\sigma^2}(x+y) + \frac{\rho}{\sigma^2(1-\rho^2)}xy\right\}$$