

## Exercise 2.3

(a)  $X_1, X_2, \dots$   $X_i \sim \text{Bi}(1, p_i)$

Show that  $X_n \xrightarrow{pr} 0$  as long as  $p_n \rightarrow 0$ , but  $X_n \xrightarrow{as} 0$  takes more

$$\lim_{n \rightarrow \infty} \Pr(|X_i - 0| > \varepsilon) = \lim_{n \rightarrow \infty} \Pr(X_i = 1) = 0 \Leftrightarrow \lim_{n \rightarrow \infty} p_n = 0$$

From Borel-Cantelli lemma,

- if  $\sum_{i=1}^{\infty} p_i < \infty$ ,  $\Pr(\text{infinitely many } 1) = 0$

- if  $\sum_{i=1}^{\infty} p_i = \infty$   $\wedge$   $X_i \perp X_j$   $i \neq j$ ,  $\Pr(\text{infinitely many } 1) = 1$

$\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$  is a Dirichlet series, that converges only if  $\alpha > 1$

Therefore:

$\alpha = 0.999$ , series diverges

$\alpha = 1$ , diverges (harmonic series)

$\alpha = 1.01$ , converges

i.e., if  $\alpha < 1$   $X_n \xrightarrow{pr} 0$ , but not almost surely

(b)  $U \sim \text{Unif}(0, 1)$

$A_1 = [0, 1/2]$ ,  $A_2 = (1/2, 1]$

$A_3 = [0, 1/4]$ ,  $A_4 = [1/4, 1/2]$ ,  $A_5 = (1/2, 3/4]$ ,  $A_6 = (3/4, 1]$

and so on

$$X_n = \mathbb{1}(U \in A_n)$$

•  $|A_n| \xrightarrow{n \rightarrow \infty} 0 \Rightarrow \lim_{n \rightarrow \infty} \Pr(|X_i \in A_n| > \varepsilon) = 0 \Rightarrow X_n \xrightarrow{pr} 0$

• but there is always one  $A_k$  among the groups for which  $X_n = 1$

$\{0, 1\}$  for  $A_1, A_2$

$\{0, 0, 0, 1\}$  for  $A_3, A_4, A_5, A_6$

and so on

$$\Rightarrow X_n \not\xrightarrow{as} 0$$

(c)  ~~$X_0 \sim \text{Unif}(0, 1)$~~

~~$$X_{2n} = X_0$$~~

~~$$X_{2n+1} = 1 - X_0$$~~

~~$$X_n \xrightarrow{as} \text{Unif}(0, 1)$$~~

~~$$\Pr(|X_{2n} - x| \geq \varepsilon) + \Pr(|X_{2n+1} - x| \geq \varepsilon)$$~~

~~$$\Pr(\max\{|X_{2n} - x|, |1 - X_{2n} - x|\} < \varepsilon)$$~~

$$(c) X_n = \begin{cases} 0 & \text{with } p = 1/2 \\ 1 & \text{" } p = 1/2 \end{cases}$$

$$X_n \xrightarrow{d} \text{Bi}(1, 1/2)$$

$$\cancel{X_{n+1}} = \begin{cases} 0 & \text{if } X_n = 1 \\ 1 & \text{if } X_n = 0 \end{cases}$$

$$\Pr(|X_n - x| > \varepsilon) + \Pr(|X_{n+1} - x| > \varepsilon) = 1 \quad X_n \not\xrightarrow{d} x$$

$$(d) X_n = \begin{cases} a_n & \text{with prob } p_n \\ 0 & \text{" } 1-p_n \end{cases}$$

$$p_n = \frac{1}{n} \quad \lim_{n \rightarrow \infty} \Pr(|X_n - 0| > \varepsilon) = 0$$

$$\lim_{n \rightarrow \infty} E[X_n] \rightarrow 1$$

$$\lim_{n \rightarrow \infty} E[X_n] = a_n p_n + (1-p_n) \cdot 0 \quad a_n = \frac{1}{p_n} \Rightarrow a_n = n$$

$$(e) X_n = \theta + Z_n, \quad \theta \in \mathbb{R}, \quad Z_1, Z_2, \dots \text{ iid with } Z_n = \begin{cases} 2^n & 1/3n \\ -2^n & 1/3n \\ 0 & 1 - \frac{2}{3n} \end{cases}$$

$$\lim_{n \rightarrow \infty} \Pr(|X_n - \theta| > \varepsilon) = 0 \Rightarrow X_n \xrightarrow{p} \theta$$

$$\lim_{n \rightarrow \infty} E[|X_n - \theta|] = \lim_{n \rightarrow \infty} E[|Z_n|] = \frac{2^n}{3n} + \frac{2^n}{3n} \neq 0$$

# Exercise 2.10

~~Let~~  $X \geq 0$ , then  $\Pr(X \geq a) \leq \frac{E[X]}{a} \quad \forall a > 0$

$$\frac{E[X]}{a} = \frac{\int_0^a x f(x) dx + \int_a^{+\infty} x f(x) dx}{a}$$

$$\geq \frac{\int_a^{+\infty} x f(x) dx}{a} \geq \frac{\int_a^{+\infty} a f(x) dx}{a}$$

$$= \int_a^{+\infty} f(x) dx = \Pr(X \geq a)$$

(a)  $E[h(x)] = \int_0^a h(x) f(x) dx + \int_a^{+\infty} h(x) f(x) dx$

$$\geq \int_a^{+\infty} h(x) f(x) dx \geq \int_a^{+\infty} h(a) f(x) dx = h(a) \Pr(X \geq a)$$

$h(x) = \frac{x^2}{a = \varepsilon} \quad E[(X - E[X])^2] \geq \Pr[(X - E[X])^2 > \varepsilon^2]$   
 $= \Pr[|X - E[X]| > \varepsilon]$  as  $X$  is non-negative

(b)  $X_1, \dots, X_n$  iid r.v. s.t.  $E[X] = \mu$ ,  $\text{Var}(X) = \sigma^2$

$$\Pr[|\bar{X}_n - \mu| \geq \varepsilon] \leq \frac{E[(\bar{X}_n - \mu)^2]}{\varepsilon^2} = \frac{\text{Var}(X)}{n \varepsilon^2} \xrightarrow{n \rightarrow \infty} 0$$

(c)  $\hat{\theta}_n = \frac{\sum w_i Y_i}{\sum w_i} \quad w_i > 0$

$$\lim_{n \rightarrow \infty} \Pr(|\hat{\theta}_n - \theta| > \varepsilon) = 0 \iff \frac{\text{Var}(\hat{\theta})}{\varepsilon^2} \rightarrow 0 \quad \text{because point (b)}$$

$\text{Var}(\hat{\theta}) = \frac{1}{(\sum w_i)^2} \sum w_i^2 \text{Var}(Y_i)$   $\lim_{n \rightarrow \infty} \frac{\sum \frac{1}{i^2}}{\sum (\frac{1}{i})^2} \rightarrow a \quad \theta(\frac{1}{n}) \Rightarrow \rightarrow 0$   
 $\lim_{n \rightarrow \infty} \frac{\sum \frac{1}{i^3}}{(\sum \frac{1}{i^{3/2}})^2} \rightarrow a \quad \theta(\frac{1}{n}) \Rightarrow \neq 0$

(d)  $X$ ,  $E[X] = \mu$  Th.  $\Pr[|X - \mu| > \varepsilon] \leq E[|X - \mu|^4] / \varepsilon^4$

$$E\left[\frac{|X - \mu|^4}{\varepsilon^4}\right] \geq \Pr[(X - \mu)^4 \geq \varepsilon^4] = \Pr[|X - \mu| \geq \varepsilon]$$

$\uparrow$   $X$  non-negative

$X \sim \mathcal{N}(\mu, \sigma^2) \rightarrow \Pr(|X - \mu| \geq \varepsilon) \leq \frac{3\sigma^4 + k_4}{\varepsilon^4} \stackrel{!}{=} 0$  (see ex 1.38)

(e)  $X_1, \dots, X_n$  iid with finite 4<sup>th</sup> moment  $\delta_4 = E\left[\left(\frac{X_i - \mu}{\sigma}\right)^4\right] - 3$

Show that  $E[|\bar{X}_n - \mu|^4] = \frac{\sigma^2}{n^2} \left(3 + \frac{1}{n} (\delta_4 - 3)\right)$

$\bar{X}_n = \frac{\sum X_i}{n}$

$$E\left[\frac{\sigma^4}{n^4} \left(\sum_{i=1}^n \frac{X_i - \mu}{\sigma}\right)^4\right] = \frac{\sigma^4}{n^4} \left( \sum_{i=1}^n E\left[\left(\frac{X_i - \mu}{\sigma}\right)^4\right] + 6 \frac{n(n-1)}{2} E\left[\left(\frac{X_i - \mu}{\sigma}\right)^2\right] E\left[\left(\frac{X_j - \mu}{\sigma}\right)^2\right] \right)$$

+ all term with  $E[X_i - \mu] = 0$

$$= \frac{\sigma^4}{n^4} (n\delta_4 + 3n + 3n^2 - 3n)$$

$$= \frac{\sigma^2}{n^2} \left(3 + \frac{\delta_4}{n}\right) \quad \text{?} - 3$$

# Exercise 2.22

(a)  $X_n \xrightarrow{d} X$ ,  $Y_n \xrightarrow{p} 0$ , then  $Y_n \xrightarrow{d} X$

$$\forall \varepsilon \lim_{n \rightarrow \infty} \Pr[|Y_n - X_n| > \varepsilon] = 0$$

$$\Pr[Y_n \leq y] = \Pr[X_n + \underbrace{Y_n - X_n}_{\xrightarrow{p} 0} \leq y] \\ = \Pr[X_n \leq y] = F_X(y)$$

(b)  $X_n \xrightarrow{d} X$ ,  $Y_n \xrightarrow{p} a$ ,  $(X_n, Y_n) \xrightarrow{d} (X, a)$

$$F_{X_n, Y_n}(x, y) = \Pr(X_n \leq x, Y_n \leq y) \\ = \Pr(X_n \leq x) \Pr(Y_n \leq a) + \Pr(X_n \leq x) \Pr(a \leq Y_n \leq y)$$

$$\text{if } (y < a), \Pr(Y_n < a) = 0$$

$$\text{if } (y \geq a), \Pr(Y_n \geq a) = 1 \Rightarrow F_{X_n, Y_n}(x, y) = F_X(x)$$

$$\Downarrow \\ (X_n, Y_n) \rightarrow (X, a)$$

(c)  $X_n \xrightarrow{d} X$  and  $Y_n \xrightarrow{p} a$ , Th:  $X_n + Y_n \xrightarrow{d} X + a$  (i)

$$X_n Y_n \xrightarrow{d} X a \quad \text{(ii)}$$

$$X_n / Y_n \xrightarrow{d} X / a \quad \text{(iii)}$$

(i) w.l.g.,  $a = 0$ , ~~the following proof is for the case where  $a \neq 0$~~

$$\forall \varepsilon > 0 \quad F_{X_n + Y_n}(x) = \Pr(X_n + Y_n \leq x) = \Pr(X_n \leq x - Y_n)$$

$$= \Pr(X_n \leq x - Y_n) \cdot \Pr(|Y_n| \geq \varepsilon) + \underbrace{\Pr(X_n \leq x - Y_n)}_{\alpha} \Pr(|Y_n| < \varepsilon)$$

$$\alpha \leq \Pr[X_n \leq x + \varepsilon] \Pr[|Y_n| < \varepsilon] \leq \Pr[X_n \leq x + \varepsilon] = F_X(x + \varepsilon)$$

$$\alpha \geq \Pr[X_n \leq x - \varepsilon] \Pr[|Y_n| < \varepsilon] \geq \underbrace{1 - \Pr[X_n \geq x - \varepsilon]}_{F_X(x - \varepsilon)} - \underbrace{\Pr[|Y_n| \geq \varepsilon]}_{\rightarrow 0}$$

$$\lim_{n \rightarrow \infty} F_{X_n}(x - \varepsilon) \leq \Pr(X_n \leq x - Y_n) \Pr(|Y_n| < \varepsilon) \leq F_X(x + \varepsilon) \rightarrow F_X(x)$$

(ii)  $X_n Y_n \xrightarrow{d} X a$

(a)  $a=0$ , by Theorem 5.5.13, we need to show  $Y_n X_n \xrightarrow{p} 0$

$$\begin{aligned} \Pr(|X_n Y_n| \geq \varepsilon) &= \Pr(|X_n Y_n| \geq \varepsilon) \cap \Pr(|Y_n| \geq \delta) + \Pr(|X_n Y_n| \geq \varepsilon) \cap \Pr(|Y_n| < \delta) \\ &\leq \Pr(|Y_n| \geq \delta) + \Pr(|X_n| \geq \varepsilon/\delta) \end{aligned}$$

$$\Rightarrow 0 \leq \Pr(|X_n Y_n| \geq \varepsilon) \leq \underbrace{\Pr(|X_n| \geq \varepsilon/\delta)}_{\rightarrow 0 \text{ as } \delta \rightarrow 0 \text{ (choose } \delta \text{ small enough)}}$$

(b)  $a \neq 0$   $X_n Y_n = a X_n + X_n (Y_n - a)$

$\xrightarrow{p} 0$  in prob.

$a > 0$   $\Pr(a X_n \leq x) = \Pr(X_n \leq x/a) \xrightarrow{\text{because } X_n \xrightarrow{d} X} \Pr(X \leq x/a) = \Pr(a X \leq x)$

$a < 0$   $\Pr(a X_n \leq x) = 1 - \Pr(a X_n > x) = 1 - \Pr(X_n < x/a)$   
 $\xrightarrow{\text{because } X_n \xrightarrow{d} X} 1 - \Pr(X < x/a) = 1 - \Pr(a X > x) = \Pr(a X \leq x)$

(iii) for Theorem 5.5.4

$Y_n \rightarrow a \Rightarrow \frac{1}{Y_n} \rightarrow \frac{1}{a}$ . Then it follows from (ii)

(d)  $X_1, \dots, X_n$  iid  $X_i \sim \text{Bi}(1, p)$

$\frac{\sqrt{n}(\bar{X}_n - p)}{\hat{\sigma}^2} \xrightarrow{d} N(0, 1)$ , where  $\hat{\sigma}^2 = \hat{p}(1 - \hat{p})$

•  $\sqrt{n}(\bar{X}_n - p) \xrightarrow{d} N(0, p(1-p))$  by the CLT  
 •  $\hat{\sigma}^2 \xrightarrow{p} p(1-p)$  consistent estimator

Because of point (c), subpoint (ii),  $\frac{\sqrt{n}(\bar{X}_n - p)}{\hat{\sigma}^2} \xrightarrow{d} N(0, 1)$

(e) it follows because matrix multiplication is a sum of products of random variables

(f) it follows from point (e)

(g)  $X \sim N(0, 1)$

$Y_n = \begin{cases} 1 + 1/n & \text{with } p = 1/2 \\ 1 - 1/n & \text{with } p = 1/2 \end{cases}$

### Exercise 2.34

$$X_1, X_2, \dots \quad M_X(t) = E[e^{tx}] \quad , \quad E[X_i] = 0, \text{Var}[X_i] = \sigma^2$$

$$(a) M_X(t) = E[e^{tx}]$$

Taylor expansion  
around 0

$$\begin{aligned} &= E\left[ e^{0t} + (x-0)t e^{t \cdot 0} + \frac{(x-0)^2}{2!} t^2 e^{t \cdot 0} + o(t^2) \right] \\ &\stackrel{0=E[X]}{=} 1 + 0 + \frac{\sigma^2}{2} t^2 + o(t^2) \end{aligned}$$

$$(b) \sqrt{n} \bar{X}_n = n^{-1/2} \sum_{i=1}^n X_i = Y$$

$$M_Y(t) = \prod_{i=1}^n M_{X_i}(t/\sqrt{n}) = M_{X_i}(t/\sqrt{n})^n \stackrel{\text{because point (a)}}{=} \left( 1 + \frac{\sigma^2}{2} \frac{t^2}{n} + o\left(\frac{t^2}{n}\right) \right)^n$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left( 1 + \frac{\sigma^2}{2} \frac{t^2}{n} + o\left(\frac{t^2}{n}\right) \right)^n \\ = e^{\frac{\sigma^2 t^2}{2}} = M_{X(0,1)}(t) \end{aligned}$$

$$\lim_{n \rightarrow \infty} \left( 1 + \frac{a}{n} \right)^n = e^a$$