

83

(a) $X_1, X_2, \dots \sim \text{Bi}(1, 1/2)$

$$Y = \sum_{j=1}^{\infty} \frac{X_j}{2^j}$$

Thesis: $E[Y] = 1/2$ $\text{Var}[Y] = 1/12$

$$E[Y] = E\left[\sum_{j=1}^{\infty} \frac{X_j}{2^j}\right] = \sum_{j=1}^{\infty} \frac{E[X_j]}{2^j} = \sum_{j=1}^{\infty} E[X_j] \left(\frac{1}{2}\right)^j \quad E[aX] = aE[X]$$

geom. series $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ if $|r| < 1$

$$\frac{1}{2} < 1 \quad \sum_{j=0}^{\infty} E[X_j] \left(\frac{1}{2}\right)^j = E[X_0] \left(\frac{1}{2}\right)^0 = \frac{1/2}{1-1/2} = \frac{1/2}{1/2} = 1/2$$

$$\text{Var}[Y] = \text{Var}\left[\sum_{j=1}^{\infty} \frac{X_j}{2^j}\right] = \sum_{j=1}^{\infty} \frac{\text{Var}[X_j]}{\left(\frac{1}{2}\right)^j} = \sum_{j=1}^{\infty} \text{Var}[X_j] \left(\frac{1}{4}\right)^j$$

$$\frac{1}{4} < 1 \quad \sum_{j=0}^{\infty} \text{Var}[X_j] \left(\frac{1}{4}\right)^j = \text{Var}[X_0] \left(\frac{1}{4}\right)^0 = \frac{1/4}{1-1/4} = \frac{1/4}{3/4} = \frac{1}{3}$$

(b) $Y_n = \sum_{j=1}^n \frac{X_j}{2^j}$

$M_X(t) = e^{-tX + \beta}$ $M_2(t) = \underline{\underline{M_X(2t)}}$

$$M_{Y_n}(t) = \prod_{j=1}^n M_{X_j}\left(\frac{t}{2^j}\right) = \prod_{j=1}^n (1 - p + pe^{t/2^j}) \quad p = \frac{1}{2}$$

$$= \prod_{j=1}^n \left(\frac{1}{2} + \frac{1}{2}e^{t/2^j}\right) = \frac{1}{2^n} \prod_{j=1}^n (1 + e^{t/2^j})$$

(c) Given $(x^{1/2^n} - 1) \prod_{j=1}^n (x^{1/2^j} + 1) = x - 1$, show that

$$M_{Y_n}(t) = \frac{e^t - 1}{2^n (e^{t/2^n} - 1)}$$

$$\begin{aligned} M_{Y_n}(t) &= \frac{1}{2^n} \prod_{j=1}^n (1 + e^{t/2^j}) \\ &= \frac{1}{2^n} \frac{(e^{t/2^n} - 1) \prod_{j=1}^n (1 + e^{t/2^j})}{e^{t/2^n} - 1} \quad x-1 \rightarrow e^t - 1 \\ &= \frac{e^t - 1}{2^n (e^{t/2^n} - 1)} \Rightarrow \text{convergence in distrib.} \end{aligned}$$

$$\lim_{n \rightarrow \infty} \frac{e^t - 1}{2^n (e^{t/2^n} - 1)} = \lim_{n \rightarrow \infty} (e^t - 1) \frac{t (1/2)^n}{t e^{(1/2)^n t} - 1} = \frac{e^t - 1}{t}$$

$\lim_{x \rightarrow 0} \frac{x^n t}{e^{x^n t} - 1}$

$\frac{e^t - 1}{t(b-a)}$
Mgf of $U \sim \text{Unif}(a, b)$
 $a=0, b=1$

Unif distrib is a continuous r.v. (not countable)

Gamma, Gaussian, Cauchy, Exponential, ...

$$F_X(x) = \Pr[X \leq x]$$

$$= \int_{-\infty}^x f_X(x) dx$$

probability density function

$$\text{mgf } M_X(t) = E[e^{tx}] = \int_{-\infty}^{+\infty} e^{tx} f_X(x) dx \quad \text{assuming that } E[e^{tx}] \exists \text{ in } (s, s), s > 0$$

(i) given a r.v. Y with density $f_Y(y)$, we are asked to show that the density of $V = \text{Log} \frac{Y}{1-Y}$ is

$$g(v) = \int \left(\frac{e^v}{1+e^v} \right) \frac{e^v}{(1+e^v)^2}$$

We need the distribution of a function of 2 r.v.

- in general, very difficult, except in special cases:
- the r.v. is discrete
- the r.v. is continuous, but g is monotone

Consider the transformation $y = g(x)$, where g is a function $X \rightarrow Y$
 X is the original sample space, Y is the sample space of the transformed r.v. Y
 $g^{-1}: Y \rightarrow X$ is the inverse mapping

Discrete case
 if X is a discrete r.v. (X countable), then

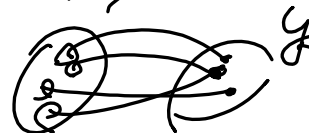
$$P_Y(y) = P[Y=y] = \sum_{x \in g^{-1}(y)} P[X=x] = \sum_{x \in g^{-1}(y)} p_X(x) \quad \text{for } y \in Y$$

otherwise

Y is also countable $Y = \{y: y = g(x), x \in X\}$

Finding the pmf of Y is just:

- identify $g^{-1}(y)$, $\forall y \in Y$
- sum the appropriate probabilities



Binomial example

$$X \sim \text{Bi}(n, \pi), \quad n \in \mathbb{N}, \quad \pi \in [0, 1]$$

$$Y = n - X, \quad y = g(x) = \underline{n - x}$$

$$\text{a) } \forall y \quad n - x = y \quad \overset{\text{one-to-one}}{\Leftrightarrow} \quad \underline{x = n - y} \quad g^{-1}(y) = n - y$$

$$P_Y(y) = \sum_{x \in g^{-1}(y)} p_X(x)$$

$$p_X(x) = \binom{n}{x} \pi^x (1-\pi)^{n-x}$$

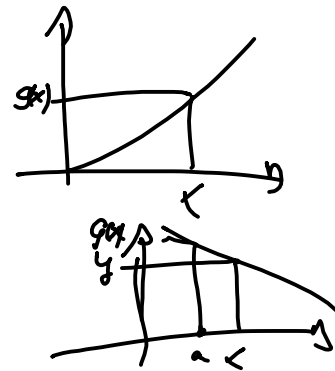
$$\begin{aligned} &= \binom{n}{n-y} \pi^{n-y} (1-\pi)^{n-(n-y)} \\ &= \binom{n}{y} \pi^{n-y} (1-\pi)^y \end{aligned}$$

$$\rightarrow Y \sim \text{Bi}(n, 1-\pi)$$

Continuous case

$$Y = g(X)$$

$$\begin{aligned} F_Y(y) &= \Pr[Y \leq y] \\ &= \Pr[g(X) \leq y] \\ &= \Pr[X \in \mathcal{X} : g(X) \leq y] \\ &= \int_{\mathcal{X} : g(x) \leq y} f_X(x) dx \end{aligned}$$



• not easy to find $\{x \in \mathcal{X} : g(x) \leq y\}$ in general, but it is possible if g is monotone

monotone $u > v \Rightarrow g(u) > g(v)$ (increasing), $u > v \Rightarrow g(u) < g(v)$ (decreasing)

If g is monotone, then $g^{-1}(x)$ is single-valued i.e. $g^{-1}(y) = x \Leftrightarrow y = g(x)$, then

$$F_Y(y) = \int_{\mathcal{X} : x \leq g^{-1}(y)} f_X(x) dx = \int_{-\infty}^{g^{-1}(y)} f_X(x) dx = F_X(g^{-1}(y)) \quad \text{if } g \text{ is increasing}$$

$$\text{or } F_Y(y) = \int_{\mathcal{X} : x \leq g^{-1}(y)} f_X(x) dx = \int_{g^{-1}(y)}^{+\infty} f_X(x) dx = 1 - F_X(g^{-1}(y)) \quad \text{if } g \text{ is decreasing}$$

Theorem 2.1.5: Let X have pdf $f_X(x)$ and $Y = g(X)$, with g monotone

Let $\mathcal{X} = \{x : f_X(x) > 0\}$ and $\mathcal{Y} = \{y : y = g(x) \text{ for some } x \in \mathcal{X}\}$

Suppose that $f_X(x)$ is continuous on $\mathcal{X}$ and $g^{-1}(y)$ has a continuous derivative in $\mathcal{Y}$

$$\text{Then } f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \cdot \left| \frac{d}{dy} g^{-1}(y) \right| & y \in \mathcal{Y} \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Proof } f_Y(y) = \frac{dF_Y(y)}{dy} = \begin{cases} \frac{dF_X(g^{-1}(y))}{dy} = f_X(g^{-1}(y)) \frac{d}{dy} g^{-1}(y) \\ \frac{d(1 - F_X(g^{-1}(y)))}{dy} = -f_X(g^{-1}(y)) \frac{d}{dy} g^{-1}(y) \end{cases}$$

Back to our story #83, part (g)

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \cdot \left| \frac{d}{dy} g^{-1}(y) \right| & y \in \mathcal{Y} \\ 0 & \text{otherwise} \end{cases}$$

$$\rightarrow g^{-1}(v) = \frac{e^v}{1+e^v}$$

$$\begin{aligned} f_Y(v) &= f_X\left(\frac{e^v}{1+e^v}\right) \left| \frac{d}{dv} \frac{e^v}{1+e^v} \right| \\ &= f_X\left(\frac{e^v}{1+e^v}\right) \frac{e^v(1+e^v) - (e^v)^2}{(1+e^v)^2} = f_X\left(\frac{e^v}{1+e^v}\right) \frac{e^v}{(1+e^v)^2} \end{aligned}$$

similarly, starting from

$$\begin{aligned} f_Y(y) &= f_X\left(\log \frac{y}{1-y}\right) \left| \frac{d}{dy} \log \left(\frac{y}{1-y}\right) \right| \\ &= f_X\left(\log \frac{y}{1-y}\right) \frac{1-y}{y} \cdot \frac{1-y+y}{(1-y)^2} = f_X\left(\log \frac{y}{1-y}\right) \frac{1}{y(1-y)} \end{aligned}$$

Theorem 2.1.5 also works 'piecewise'

Theorem 2.1.8 Let X have pdf $f_X(x)$, $Y = g(X)$, $\mathcal{Y}$ defined as in Th. 2.1.5 s.t.
we can partition $\mathcal{X}$ in $A_0, A_1, \dots, A_k$ s.t. $P_X(X \in A_0) = 0$
 $f_X(x)$ is continuous in A_i $\forall i=1, \dots, k$. Suppose that $\exists g_1(x), \dots, g_k(x)$, s.t.

- i) $g(x) = g_i(x) \quad \forall x \in A_i$
- ii) $g_i(x)$ is monotone in each A_i
- iii) $\mathcal{Y} = \{y : y = g(x), \text{ for some } x \in \mathcal{X}\} \quad \forall A_i, i=1, \dots, k$
- iv) $g^{-1}(y)$ has^a continuous derivative on $\mathcal{Y}$

Then

$$f_Y(y) = \begin{cases} \sum_{i=1}^k f_X(g_i^{-1}(y)) \left| \frac{d}{dy} g_i^{-1}(y) \right| & y \in \mathcal{Y} \\ 0 & \text{otherwise} \end{cases}$$

- g can be monotone piecewise

Theorem 2.1.10 If X has continuous cdf $F_X(x)$ and $Y = F_X(X)$,
 then $Y \sim \text{Unif}(0,1)$, i.e. $\Pr[Y \leq y] = y$

Proof

$$\begin{aligned} \Pr[Y \leq y] &= \Pr[F_X(X) \leq y] \\ &= \Pr[F^{-1}(F_X(X)) \leq F^{-1}(y)] \quad \text{as } F^{-1}(y) \text{ is increasing} \\ &= \Pr[X \leq F^{-1}(y)] \\ &= F_X(F^{-1}(y)) = y \end{aligned}$$

Location and scale families

- families of ~~distributions~~ with specific physical characteristics
- constructed by transforming of specific pdf $f_X(x)$, called standard pdf, in a specific way

Theorem 3.5.1 Let $f(x)$ be any pdf, and $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}^+$

Then $g(x; \mu, \sigma) = \frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right)$ is a pdf

Proof we need to show that $\frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right) \geq 0$ and $\int_{-\infty}^{\infty} \frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right) dx = 1$

1) since $f(x)$ is a pdf, $f(x) \geq 0 \Rightarrow g(x; \mu, \sigma) \geq 0 \quad \forall x$

2) $\int_{-\infty}^{\infty} \frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right) dx \quad y = \frac{x-\mu}{\sigma}, \int_{-\infty}^{\infty} f(y) dy = 1$

• LOCATION FAMILIES: $f(x-\mu)$ with $f(x)$ a pdf, $\mu \in \mathbb{R}$

- shift the distribution, shape maintained

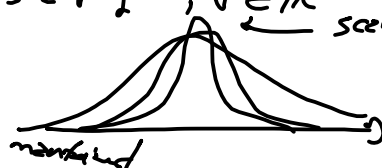
- e.g., $N(\mu, \sigma)$ from $N(0, \sigma)$



• SCALE FAMILIES: $\frac{1}{\sigma} f\left(\frac{x}{\sigma}\right)$, $f(x)$ is a pdf, $\sigma \in \mathbb{R}^+$

- e.g., $N(\mu, \sigma)$ from $N(\mu, 1)$

- stretch (or contract) the distribution, shape maintained



• LOCATION-SCALE FAMILIES: $\frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right)$ with $f(x)$ pdf, $\mu \in \mathbb{R}$, $\sigma \in \mathbb{R}^+$

- stretch/contract and shift the dist., shape kept

- e.g., $N(\mu, \sigma)$ from $N(0, 1)$