

Theorem 3.5.6 Let f be a pdf, $\mu \in \mathbb{R}$, $\sigma^2 \in \mathbb{R}_+$, Then X is a r.v. with pdf $\frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right)$ IFF

$\exists$ r.v. Z with pdf $f_Z(z)$ and $X = \sigma Z + \mu$

~~Proof~~
 IF $g(z) = \sigma z + \mu$ $x = g(z)$ $\frac{dg'(x)}{dx} = \frac{1}{\sigma}$
 g is monotone, $g'(z) = \frac{x-\mu}{\sigma} \rightarrow$
 by Theorem 2.1.5 $f_X(x) = f_Z(g^{-1}(x)) \left| \frac{dg^{-1}(x)}{dx} \right| = \underline{f\left(\frac{x-\mu}{\sigma}\right) \frac{1}{\sigma}}$

ONLY IF $g^{-1}(x) = \frac{x-\mu}{\sigma}$ $g'(z) = \sigma z + \mu$ $\frac{d'g'(z)}{dz} = \sigma$
 $f(z) = f(g^{-1}(z)) \left| \frac{dg^{-1}(z)}{dz} \right| = \frac{1}{\sigma} f\left(\frac{\sigma z + \mu - \mu}{\sigma}\right) \sigma = f(z)$

Moreover $\sigma Z + \mu = \sigma g(x) + \mu = \sigma \frac{x-\mu}{\sigma} + \mu = \underline{x}$

Important case: $\mu=0, \sigma=1$ $f_Z(z) = f\left(\frac{z-0}{1}\right) = f(z)$

$$E[X] = \underbrace{\sigma}_{=1} E[\underbrace{Z}_{=x}] + \underbrace{\mu}_{=0} = \underline{E[X]}$$

Exercise 1.56

- about exponential families

If a pdf/pmf can be written in the form

$$f(x; \theta) = \underbrace{h(x)} c(\theta) \exp \left\{ \sum \underbrace{w_i(\theta)} \underbrace{t_i(x)} \right\}$$

$$\Rightarrow \underbrace{h(x)} \exp \left\{ \sum \underbrace{w_i(\theta)} \underbrace{t_i(x)} - \underbrace{\log c(\theta)}_{\eta(\theta)} \right\}$$

then the pdf/pmf belongs to the exponential family

Here

$$h(x) \geq 0$$

$t_1(x), \dots, t_k(x)$ are real functions of x

$c(\theta) \geq 0$, standardization

$w_1(\theta), \dots, w_k(\theta)$ are real value functions of $\theta = (\theta_1, \dots, \theta_k)$

Examples

• Gaussian $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left\{ -\frac{1}{2} \frac{(x-\mu)^2}{\sigma^2} \right\}$

$$h(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left\{ -\frac{\mu^2}{2\sigma^2} \right\} \cdot \exp \left\{ \underbrace{\frac{1}{\sigma^2} \left(-\frac{x^2}{2} \right)}_{w_1(\mu, \sigma^2)} + \underbrace{\frac{\mu}{\sigma^2} x}_{w_2(\mu, \sigma^2)} \right\} t_1(x)$$

$$k=2 \quad \underbrace{(\mu, \sigma^2)}_{k=2} \cdot \theta$$

• Exp

$$f(x; \lambda) = \begin{cases} \frac{1}{\lambda} \exp \left\{ -\frac{x}{\lambda} \right\} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$= \underbrace{\frac{1}{\lambda}}_{h(x)} \underbrace{\frac{1}{\lambda}}_{c(\lambda)} \exp \left\{ -\frac{1}{\lambda} x \right\} \underbrace{1}_{w(\lambda)} t(x)$$

$$k=1$$

on the boundary

$$f(x; \lambda) = \begin{cases} \frac{1}{\lambda} \exp \left\{ 1 - \frac{x}{\lambda} \right\} & \text{if } x > \lambda \\ 0 & \text{otherwise} \end{cases}$$

$$= \underbrace{\frac{1}{\lambda}}_{h(x)} \underbrace{\frac{1}{\lambda}}_c \exp \left\{ -\frac{x}{\lambda} \right\} \underbrace{1}_{w(\lambda)} t(x)$$

$$\underline{h(x)} c(\theta) \exp(w(\theta) t(x))$$

Theorem 3.4.2 if X is a pdf/pmf in the exp. family, then

$$E\left[\sum_{i=1}^n \frac{\partial w_i(\theta)}{\partial \theta_j} t_i(X)\right] = -\frac{\partial}{\partial \theta_j} \log c(\theta) \rightarrow \frac{\partial}{\partial \theta_j} k(\theta)$$

$$Var\left[\sum_{i=1}^n \frac{\partial w_i(\theta)}{\partial \theta_j} t_i(X)\right] = -\frac{\partial^2}{\partial \theta_j^2} \log c(\theta) = E\left[\sum_{i=1}^n \frac{\partial^2 w_i(\theta)}{\partial \theta_j^2} t_i(X)\right]$$

Example $X \sim \text{Exp}(\lambda)$ $f_X(x; \lambda) = \underbrace{\frac{1}{(0, \infty)} \lambda}_{h(x)} \underbrace{\frac{1}{\lambda}}_{c(\lambda)} \exp\left\{-\underbrace{\frac{1}{\lambda} x}_{w(\theta, \lambda)}\right\} \underbrace{t(x)}_{t(x)}$

$$E\left[\frac{1}{\lambda^2} X\right] = \frac{1}{\lambda} \rightarrow \frac{1}{\lambda} E[X] = \frac{1}{\lambda} \quad E[X] = \lambda + \frac{\log \lambda}{\lambda}$$

$$Var\left[\frac{1}{\lambda^2} X\right] = -\frac{1}{\lambda^2} + E\left[\frac{2}{\lambda^3} X\right] \quad \frac{1}{\lambda}$$

$$\frac{1}{\lambda^3} Var[X] = -\frac{1}{\lambda^2} + \frac{2\lambda}{\lambda^3} \rightarrow \frac{1}{\lambda^3} Var[X] = \frac{1}{\lambda^2} \quad \frac{1}{\lambda^2}$$

$$X \sim \text{Bi}(n, \pi)$$

$$P(x) = \binom{n}{x} \pi^x (1-\pi)^{n-x} \quad x \log \pi + (n-x) \log(1-\pi)$$

$$= \frac{1}{(0, \dots, n)} \binom{n}{x} (1-\pi)^n \exp\left\{\log \frac{\pi}{1-\pi} x\right\}$$

$$E\left[\frac{1-\pi}{n} \frac{(1-\pi)^{n-x}}{(1-\pi)^n} x\right] = \frac{n}{1-\pi}$$

$$\frac{1}{\pi(1-\pi)} E[X] = \frac{n}{1-\pi} \quad E[X] = n\pi$$

$$- \frac{n}{1-\pi} + \frac{n}{(1-\pi)^2}$$

$$Var\left(\frac{1}{\pi(1-\pi)} X\right) = +\frac{n}{(1-\pi)^2} - E\left[-\frac{1-2\pi}{\pi^2(1-\pi)^2} X\right]$$

$$\frac{1}{\pi^2(1-\pi)^2} Var(X) = -\frac{n}{(1-\pi)^2} + n\pi \left(\frac{1-2\pi}{\pi^2(1-\pi)^2}\right)$$

$$Var(X) = n\pi^2 + n\pi - 2n\pi^2 = n\pi(1-\pi)$$

Natural exponential families
 We can reparametrise s.t. $\eta = (\eta_1, \dots, \eta_k) = (w_1(\theta), \dots, w_k(\theta))$. Then

$$f(x, \eta) = h(x) c^*(\eta) \exp \left\{ \sum_{i=1}^k \eta_i t_i(x) \right\}$$

$$c^*(\eta) = \left[\int_{-\infty}^{+\infty} h(x) \exp \left\{ \sum_{i=1}^k \eta_i t_i(x) \right\} dx \right]^{-1}$$

Example

• $X \sim \text{Bin}(n, \pi)$ $p_x(x, \pi) = \frac{1}{\{0, \dots, n\}} \binom{n}{x} (1-\pi)^n \exp \left\{ \log \frac{\pi}{1-\pi} x \right\}$

$w(\pi) = \log \frac{\pi}{1-\pi} = \eta \Leftrightarrow \pi = \frac{e^\eta}{1+e^\eta}$ $\pi \in (0, 1)$
 $\eta \in (-\infty, +\infty)$

$p_x(x, \pi) = \frac{1}{\{0, \dots, n\}} \binom{n}{x} \left(\frac{e^\eta}{1+e^\eta} \right)^x \exp \{ \eta x \}$

• $X \sim \text{Gaussian}$ $f(x, \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left\{ -\frac{x^2}{2\sigma^2} \right\} \exp \left\{ \frac{1}{\sigma^2} \left(-\frac{x^2}{2} \right) + \frac{\mu}{\sigma^2} x \right\}$

$w_1(\mu, \sigma^2) = \frac{1}{\sigma^2} = \eta_1$

$\Theta = \mathbb{R} \times \mathbb{R}_+$

$w_2(\mu, \sigma^2) = \frac{\mu}{\sigma^2} = \eta_2$

$\mathcal{H} = \mathbb{R}_+ \times \mathbb{R}$

$f(x, \eta_1, \eta_2) = \frac{\eta_1^{1/2}}{\sqrt{2\pi}} \exp \left\{ -\frac{\eta_2^2}{\eta_1} \right\} \exp \left\{ \eta_1 \left(-\frac{x^2}{2} \right) + \eta_2 x \right\}$

Curved exponential families

if $d < k$, we have a curved exponential family

Example: Gaussian with $\sigma^2 = \mu^2$ $\dim \Theta = 1$

$f(x, \mu) = \frac{1}{\sqrt{2\pi\mu^2}} \exp \left\{ -\frac{(x-\mu)^2}{2\mu^2} \right\}$

$= \frac{1}{\sqrt{2\pi}} \frac{1}{\mu} \exp \left\{ \frac{\mu^2}{2} \right\} \exp \left\{ -\frac{x^2}{2\mu^2} + \frac{x}{\mu} \right\}$

$$f_{x,y}(x,y) = f_x(x) f_y(y) \stackrel{\text{independence}}{\iff} f(x,y) = h(x) g(y)$$

Bivariate transformation

Consider (X,Y) r.v. with joint pdf/pmf $f_{x,y}(x,y)$

$$(U,V) \text{ r.v. : } U = g_1(X,Y), V = g_2(X,Y)$$

A is the support of X,Y

$$B = \{(u,v) : u = g_1(x,y), v = g_2(x,y), \text{ for some } (x,y) \in A\}$$

Discrete case

Both A and B are countable

$$A_{uv} = \{(x,y) : u = g_1(x,y), v = g_2(x,y)\}$$

$$\text{pmf: } P_{uv}(u,v) = \Pr[U=u, V=v] = \Pr[(X,Y) \in A_{uv}] = \sum_{(x,y) \in A_{uv}} P_{x,y}(x,y)$$

Example (again show that $\text{Poisson}(A) + \text{Poisson}(B) = \text{Poisson}(\lambda + \theta)$)

$$A = \{(x,y), x=0,1,\dots, y=0,1,\dots\}$$

$$\begin{array}{lll} U = x+y & \rightarrow & x = u-v \\ V = y & & y = v \end{array} \quad \begin{array}{l} g_1(x,y) = x+y \\ g_2(x,y) = y \end{array}$$

$$B = \{(u,v), u=0,1,\dots, v=0,1,\dots\}$$

$$P_{u,v}(u,v) = P_{x,y}(u-v,v) = \frac{\lambda^{u-v}}{(u-v)!} e^{-\lambda} \frac{\theta^v}{v!} e^{-\theta}$$

$$\begin{aligned} P_u(u,v) &= \sum_{v=0}^u \frac{\lambda^{u-v}}{(u-v)!} e^{-\lambda} \frac{\theta^v}{v!} e^{-\theta} = e^{-(\lambda+\theta)} \left(\sum_{v=0}^u \frac{\lambda^{u-v} \theta^v}{(u-v)! v!} \cdot \frac{u!}{u!} \right) \frac{u!}{(u-v)! v!} = \binom{u}{v} \\ &= \frac{e^{-(\lambda+\theta)}}{u!} (\lambda+\theta)^u \sim \text{Poisson}(\lambda+\theta) \end{aligned}$$

Continuous case
same as in the univ. case, $|J|$ instead of the deriv.

$$f_{U,V}(u,v) = f_{X,Y}(h_1(u,v), h_2(u,v)) |J|$$

where
 $h_1(u,v) = x$
 $h_2(u,v) = y$

$$J = \begin{vmatrix} \frac{\partial h_1(u,v)}{\partial u} & \frac{\partial h_1(u,v)}{\partial v} \\ \frac{\partial h_2(u,v)}{\partial u} & \frac{\partial h_2(u,v)}{\partial v} \end{vmatrix}$$

$$= \left| \frac{\partial h_1(u,v)}{\partial u} \cdot \frac{\partial h_2(u,v)}{\partial v} - \frac{\partial h_1(u,v)}{\partial v} \cdot \frac{\partial h_2(u,v)}{\partial u} \right|$$

Example $X \sim \text{Gamma}(\alpha_1, \beta)$, $Y \sim \text{Gamma}(\alpha_2, \beta)$ $X \perp Y$

$$f_{X,Y}(x,y) = \frac{1}{\beta^{\alpha_1+\alpha_2} \Gamma(\alpha_1) \Gamma(\alpha_2)} x^{\alpha_1-1} y^{\alpha_2-1} \exp^{-\frac{x+y}{\beta}} \quad x > 0, y > 0$$

$$U = X+Y$$

$$V = \frac{X}{X+Y}$$

$$B = \{(u,v), u > 0, 0 < v < 1\}$$

$\begin{matrix} x = uv \\ y = u(1-v) \end{matrix}$

$$f_{U,V}(u,v) = f_{X,Y}(uv, u(1-v)) |J|$$

$$= \frac{1}{\beta^{\alpha_1+\alpha_2} \Gamma(\alpha_1) \Gamma(\alpha_2)} u^{\alpha_1-1} v^{\alpha_1-1} u^{\alpha_2-1} (1-v)^{\alpha_2-1} e^{-\frac{uv+u(1-v)}{\beta}} \begin{vmatrix} v & u \\ 1-v & -u \end{vmatrix}$$

$$= \frac{1}{\beta^{\alpha_1+\alpha_2} \Gamma(\alpha_1) \Gamma(\alpha_2)} u^{\alpha_1+\alpha_2-1} v^{\alpha_1-1} (1-v)^{\alpha_2-1} e^{-\frac{u}{\beta}} \begin{vmatrix} v & u \\ 1-v & -u \end{vmatrix}$$

$$= \underbrace{\frac{1}{\beta^{\alpha_1+\alpha_2} \Gamma(\alpha_1) \Gamma(\alpha_2)}}_{\text{Gamma}(\alpha_1+\alpha_2, \beta)} \underbrace{u^{\alpha_1+\alpha_2-1} v^{\alpha_1-1} (1-v)^{\alpha_2-1}}_{\text{Beta}(\alpha_1, \alpha_2)} e^{-\frac{u}{\beta}}$$

Note that $f_{U,V}(u,v) = f_U(u) f_V(v) \Rightarrow U \perp V$