

2.50)

a) $\sqrt{n}(A_n - a) \xrightarrow{d} N(0, 1)$

$g(a) = a^3 \rightarrow g'(a) = 3a^2$

$\sqrt{n}(A_n^3 - a^3) \xrightarrow{d} 3a^2 N(0, 1) = N(0, 9a^4)$

$g(a) = e^{ka} \quad g'(a) = k e^{ka}$

$\sqrt{n}(e^{kA_n} - e^{ka}) \xrightarrow{d} k e^{ka} N(0, 1) = N(0, k^2 e^{2ka})$

b) For Binomial $\hat{p} = \frac{X_n}{n}$

$\sqrt{n}(\hat{p} - p) \xrightarrow{d} N(0, p(1-p))$

i) $g(p) = \frac{p}{1-p} \quad g'(p) = \frac{1(1-p) - p(-1)}{(1-p)^2} = \frac{1-p+p}{(1-p)^2} = \frac{1}{(1-p)^2}$

$\sqrt{n}\left(\frac{\hat{p}}{1-\hat{p}} - \frac{p}{1-p}\right) \xrightarrow{d} \frac{1}{(1-p)^2} N(0, p(1-p)) = N(0, \frac{p(1-p)}{(1-p)^3}) = N(0, \frac{p}{(1-p)^3})$

ii) $g(p) = \ln(p) - \ln(1-p)$

$g'(p) = \frac{1}{p} + \frac{1}{1-p} = \frac{(1-p) + p}{p(1-p)} = \frac{1}{p(1-p)}$

$\sqrt{n}\left(\ln\left(\frac{\hat{p}}{1-\hat{p}}\right) - \ln\left(\frac{p}{1-p}\right)\right) \xrightarrow{d} \frac{1}{p(1-p)} N(0, p(1-p)) = N(0, \frac{1}{p(1-p)})$

iii) $\gamma(p) = 2 \sin^{-1}(\sqrt{p}) \quad \frac{d}{dx} \sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$

$\gamma'(p) = \frac{2}{\sqrt{1-p}} \cdot \frac{1}{2\sqrt{p}} = \frac{1}{\sqrt{1-p}\sqrt{p}} = \frac{\sqrt{1-p}\sqrt{p}}{(1-p)p}$

$\sqrt{n}(\gamma(\hat{p}) - \gamma(p)) \xrightarrow{d} \frac{\sqrt{1-p}\sqrt{p}}{(1-p)p} N(0, p(1-p)) = N(0, \frac{(1-p)p(1-p)}{(1-p)^2 p^2}) = N(0, 1)$

c) i) Multivariate delta method:

$\sqrt{n}(g(\hat{p}_1, \hat{p}_2) - g(p_1, p_2)) \xrightarrow{d} N(0, \Sigma)$

$g(p_1, p_2) = \frac{p_1}{p_2}$

Independence we hope, covariance is not fun. Simplifies:

$\frac{\partial g}{\partial p_1} = \frac{1}{p_2} \quad \frac{\partial g}{\partial p_2} = -\frac{p_1}{p_2^2}$

$\sqrt{n}(g(\hat{p}_1, \hat{p}_2) - g(p_1, p_2)) \xrightarrow{d} \frac{\partial g}{\partial p_1} N(0, p_1(1-p_1)) + \frac{\partial g}{\partial p_2} N(0, p_2(1-p_2))$

$= \frac{1}{p_2} N(0, p_1(1-p_1)) + \left(-\frac{p_1}{p_2^2}\right) N(0, p_2(1-p_2))$

$= N(0, \frac{p_1(1-p_1)}{p_2^2}) + N(0, \frac{p_1^2(1-p_2)}{p_2^3})$

$= N(0, \frac{p_1(1-p_1)}{p_2^2}) + N(0, \frac{(1-p_2)p_1^2}{p_2^3})$

$N(\hat{z}_1, \sigma_1^2) + N(\hat{z}_2, \sigma_2^2) = N(\hat{z}_1 + \hat{z}_2, \sigma_1^2 + \sigma_2^2)$

$= N(0, \frac{p_1(1-p_1)}{p_2^2} + \frac{p_1^2(1-p_2)}{p_2^3})$

$= N(0, \frac{p_1 p_2 (1-p_1) + p_1^2 (1-p_2)}{p_2^3})$

$= N(0, \frac{p_1(p_2(1-p) + p_1(1-p_2))}{p_2^3})$

ii) Hattify the variance to find a reasonable estimator:

$\sqrt{n}\left(\frac{\hat{p}_1}{\hat{p}_2} - \frac{p_1}{p_2}\right) \approx N(0, \dots)$

$\frac{\hat{p}_1}{\hat{p}_2} - \frac{p_1}{p_2} \approx \frac{1}{\sqrt{n}} N(0, \dots)$

$\frac{\hat{p}_1}{\hat{p}_2} \approx \frac{1}{\sqrt{n}} N(0, \dots) + \frac{p_1}{p_2}$

Variance is undisturbed by constants

$\text{Var}\left(\frac{\hat{p}_1}{\hat{p}_2} - \frac{p_1}{p_2}\right) = \text{Var}\left(\frac{\hat{p}_1}{\hat{p}_2}\right)$

$\text{Var}\left(\frac{\hat{p}_1}{\hat{p}_2}\right) = \frac{1}{n} \frac{\hat{p}_1}{\hat{p}_2^3} (\hat{p}_2(1-\hat{p}_1) + \hat{p}_1(1-\hat{p}_2))$

The mean is disturbed by constant.

$\frac{\hat{p}_1}{\hat{p}_2} \approx N\left(\frac{p_1}{p_2}, \frac{1}{n} \frac{\hat{p}_1}{\hat{p}_2^3} (\hat{p}_2(1-\hat{p}_1) + \hat{p}_1(1-\hat{p}_2))\right)$

iii) For unequal sample sizes, we couldn't have added the variances during delta method.

We would need to drag n_i with instead of factoring out:

$\text{Var} = \frac{\hat{p}_1(1-\hat{p}_1)}{n_1 \hat{p}_1^2} + \frac{\hat{p}_1^2(1-\hat{p}_2)}{n_2 \hat{p}_1^3}$

here

d) i) $Y \sim \text{Geom}\left(\frac{1}{p}, \frac{1-p}{p}\right)$

CLT $\rightarrow \sqrt{n}(\bar{Y} - \frac{1}{p}) \xrightarrow{d} N(0, \frac{1-p}{p^2})$

ii) Delta method $a = \frac{1}{p}$

$g(a) = \frac{1}{a} \quad g'(a) = -\frac{1}{a^2} = -p^2$

$\sqrt{n}(\hat{p} - p) \xrightarrow{d} -p^2 N(0, \frac{1-p}{p^2})$

$= N(0, \frac{p^4(1-p)}{p^2}) = N(0, p^2(1-p))$

e) i) To find confidence interval, harvest the normal distribution for scalars until you have a Z-distribution ($N(0, 1)$)

Then check as normal:

$P\left(\bar{Z}_{\frac{\alpha}{2}} < \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < Z_{\frac{\alpha}{2}}\right) = 1 - \alpha$

$P\left(\mu + \frac{\sigma}{\sqrt{n}} \bar{Z}_{\frac{\alpha}{2}} < \bar{X} < \mu + \frac{\sigma}{\sqrt{n}} Z_{\frac{\alpha}{2}}\right) = 1 - \alpha$

CI = [lower b, upper b]

ii) $\sqrt{n}(A_n - a) \xrightarrow{d} N(0, 1)$

$\sqrt{n}(B_n - b) \xrightarrow{d} N(0, 1)$

Circle: $X^2 + Y^2 = r^2$

$n(A_n - a)^2 + n(B_n - b)^2 = N_1^2 + N_2^2$

$(A_n - a)^2 + (B_n - b)^2 = \frac{\chi_2^2}{n}$

$P(\chi_2^2 \leq C) = 1 - \alpha$

$(A_n - a)^2 + (B_n - b)^2 \leq \frac{1}{n} C$

Lookup $\rightarrow 4, 605$

f) $\sqrt{n}(R_n - r)$

Multivariate delta method:

$g(a, b) = \sqrt{a^2 + b^2} \quad \frac{\partial g}{\partial a} = \frac{2a}{2\sqrt{a^2 + b^2}} = \frac{a}{\sqrt{a^2 + b^2}} = \frac{a}{r}$

$\frac{\partial g}{\partial b} = \frac{b}{r}$

$\sqrt{n}(R_n - r) \xrightarrow{d} \frac{a}{r} N_1 + \frac{b}{r} N_2$

$= N(0, \frac{a^2}{r^2}) + N(0, \frac{b^2}{r^2})$

$= N(0, \frac{a^2 + b^2}{r^2}) = N(0, 1)$

$\sqrt{n}(\hat{\alpha} - \alpha) \xrightarrow{d} ?$

$g(a, b) = \tan^{-1}\left(\frac{b}{a}\right) \quad \frac{d}{dx} \tan^{-1}(x) = \frac{1}{x^2 + 1}$

$\frac{\partial g}{\partial a} = \frac{1}{\frac{b^2}{a^2} + 1} \cdot -\frac{b}{a^2} = -\frac{b}{a^2 + b^2} = -\frac{b}{r^2}$

$\frac{\partial g}{\partial b} = \frac{1}{\frac{b^2}{a^2} + 1} \cdot \frac{1}{a} = \frac{1}{\frac{b^2}{a^2} + a} = \frac{a}{r^2}$

$\sqrt{n}(\hat{\alpha} - \alpha) \rightarrow -\frac{b}{r^2} N_1 + \frac{a}{r^2} N_2$

$= N(0, \frac{b^2}{r^4}) + N(0, \frac{a^2}{r^4})$

$= N(0, \frac{a^2 + b^2}{r^4}) = N(0, \frac{1}{r^2})$

$\text{Cov}(r, \alpha) = E(r\alpha) - E(r)E(\alpha)$

$E(r) = E\left(\frac{a}{r} N_1 + \frac{b}{r} N_2\right)$

$= \frac{a}{r} E(N_1) + \frac{b}{r} E(N_2)$

$= 0 \rightarrow$ who cares about $E(\alpha)$ then...

$E(r\alpha) = E\left(\left(\frac{a}{r} N_1 + \frac{b}{r} N_2\right) \left(-\frac{b}{r^2} N_1 + \frac{a}{r^2} N_2\right)\right)$

$E\left(\frac{a}{r} \cdot \left(-\frac{b}{r^2}\right) N_1^2 + \frac{b}{r} \cdot \frac{a}{r^2} N_2^2 + \frac{a}{r} \cdot \frac{a}{r^2} N_1 N_2 + \frac{b}{r} \cdot \left(-\frac{b}{r^2}\right) N_2 N_1\right)$

$E(N_1 N_2) = E(N_1)E(N_2) = \text{indep.}$

$= 0$

$E(N_1^2) = E(N_1)^2 + \text{Var}(N_1) = 1$

$E(N_2^2) = 1$

$E(r\alpha) = \frac{a}{r} \cdot -\frac{b}{r^2} \cdot 1 + \frac{b}{r} \cdot \frac{a}{r^2} \cdot 1$

$= -\frac{ab}{r^3} + \frac{ab}{r^3} = 0$

$\text{Cov}(r, \alpha) = 0 \rightarrow \text{indep.}$

g) i) $(4, 44 - a)^2 + (2, 22 - b)^2 = \frac{4,605}{100}$

$r^2 \rightarrow r \approx 0,215$

ii) $r = \sqrt{4,44^2 + 2,22^2} \approx 4,96$

$\alpha = \tan^{-1}\left(\frac{2,22}{4,44}\right) \approx 0,46$

$\bar{Z}_{\frac{\alpha}{2}} \leq \sqrt{n}(R_n - r) \leq Z_{\frac{\alpha}{2}} \Rightarrow \left[\frac{1}{\sqrt{n}} \bar{Z}_{\frac{\alpha}{2}}, \frac{1}{\sqrt{n}} Z_{\frac{\alpha}{2}}\right]$

$\bar{Z}_{\frac{\alpha}{2}} \leq r \sqrt{n}(\hat{\alpha} - \alpha) \leq Z_{\frac{\alpha}{2}} \Rightarrow \left[\frac{1}{r\sqrt{n}} \bar{Z}_{\frac{\alpha}{2}}, \frac{1}{r\sqrt{n}} Z_{\frac{\alpha}{2}}\right]$

iii) $(R_n - r)^2 + r^2(\hat{\alpha} - \alpha)^2 = \frac{1}{n} \chi_2^2$

Ellipse term χ_2^2

$\rightarrow$ same radius but elliptic.