
STK 4011-9011 Autumn 2026

Statistical Inference Theory: the Oblig

This is the Oblig, the mandatory assignment, for STK 4011-9011, Statistical Inference Theory, Autumn 2026. It is made available at the course website Monday October 12, and the submission deadline is Monday October 26, 13:58, *via the Canvas system*. Reports may be written in nynorsk, bokmål, riksmål, English, or Latin, should preferably be text-processed (for instance with TeX or LaTeX), and must be submitted as a single pdf file. The submission must contain your name & the course code (i.e. STK 4011 or STK 9011). The Oblig set contains three exercises and comprises five pages, including a one-page appendix (in addition to the present introduction page, ‘page 0’).

It is expected that you give a clear presentation with all necessary explanations, but write concisely (in der Beschränkung zeigt sich erst der Meister; brevity is the soul of wit; краткость – сестра таланта). Remember to include all relevant plots and figures. These should preferably be placed inside the text, close to the relevant subquestion.

For a few of the questions setting up an appropriate computer programme might be part of your solution. The code ought to be handed in along with the rest of the written assignment; you might place the code in an appendix.

All aids, including collaboration, are allowed, but the submission must be written by you (as in ‘you by yourself’) and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

Importantly, I ask you to write up half a page or so to tell us *how the work proceeded*: what was easy; what was not so easy; what was perhaps in the fun category; if you were stuck, how did you unstuck yourself; *how satisfied are you* with your work. And, since this is 2026 A.D., in the middle of the Manhattan Project of Our Time: did you use AI, and, in that case, how.

Application for postponed delivery: If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (email: studieinfo@math.uio.no) well before the deadline.

The mandatory assignment in this course must be approved, in the same semester, before you are allowed to take the final examination.

Complete guidelines about delivery of mandatory assignments, along with a ‘log on to Canvas’, can be found here:

www.uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

Enjoy [imperative pluralis].

Nils Lid Hjort

1. When probabilities are random

SOMETIMES EVEN PROBABILITIES have distributions. Let X, Y, Z be independent unit exponentials, i.e. with densities $\exp(-x)$, $\exp(-y)$, $\exp(-z)$ on the positive half-line. Define from these

$$S = X + Y + Z, \quad (A, B, C) = \left(\frac{X}{S}, \frac{Y}{S}, \frac{Z}{S} \right).$$

- Use moment-generating functions to establish that $S \sim \text{Gamma}(3, 1)$. Set up the joint density for (X, Y, Z) . Solve the equations to express (X, Y, Z) in terms of (S, A, B) . Use this to find the joint density $g(s, a, b)$ for (S, A, B) . Deduce from this again that (A, B) has a flat density on its outcome space, the $\{(a, b): a > 0, b > 0, a + b < 1\}$ triangle.
- Find the means and variances for A, B, C , and their covariances and correlations.
- Extend calculations above to say ten random probabilities, and comment, perhaps also on further generalisations.

2. Skewness, kurtosis, checking normality

THE WORLD ISN'T ENTIRELY NORMAL, and we shall build a little machinery for detecting this. For any distribution F , with finite fourth moment, define

$$\mathbb{E} X = \xi, \quad \text{Var } X_i = \sigma^2, \quad \text{skew} = \mathbb{E} \left(\frac{X - \xi}{\sigma} \right)^3, \quad \text{kurt} = \mathbb{E} \left(\frac{X - \xi}{\sigma} \right)^4 - 3,$$

in which X is a variable having that distribution. These quantities are the mean, the variance, the skewness, the kurtosis.

- Show that the skewness and kurtosis are independent of shift and scale, i.e. that $Y = a + bX$ has the same skewness and kurtosis as does X . Show next that the normal distribution has skewness and kurtosis equal to zero (which is why one subtracts the '3' in the definition above).
- Consider $X_1, \dots, X_n$ independent standard normal. In addition to our eternal sample mean companion $\bar{X}_n = (1/n) \sum_{i=1}^n X_i$, define the next sample averages

$$\begin{aligned} M_{n,2} &= (1/n) \sum_{i=1}^n X_i^2, \\ M_{n,3} &= (1/n) \sum_{i=1}^n X_i^3, \\ M_{n,4} &= (1/n) \sum_{i=1}^n X_i^4. \end{aligned}$$

Show that the four converge in probability to 0, 1, 0, 3, respectively. Use the multidimensional Central Limit Theorem to show that

$$\begin{pmatrix} \sqrt{n}\bar{X}_n \\ \sqrt{n}(M_{n,2} - 1) \\ \sqrt{n}M_{n,3} \\ \sqrt{n}(M_{n,4} - 3) \end{pmatrix} \rightarrow_d \begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{pmatrix} \sim N_4(0, \Sigma), \quad \text{where } \Sigma = \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 2 & 0 & 12 \\ 3 & 0 & 15 & 0 \\ 0 & 12 & 0 & 96 \end{pmatrix}.$$

(c) Show from $\sqrt{n}\bar{X}_n \rightarrow_d V_1$ that $\sqrt{n}\bar{X}_n^2 \rightarrow_{\text{pr}} 0$. Use this, along with

$$\hat{\sigma}^2 = A_{n,2} = (1/n) \sum_{i=1}^n (X_i - \bar{X}_n)^2 = M_{n,2} - \bar{X}_n^2,$$

to show that

$$\sqrt{n}(A_{n,2} - 1) \rightarrow_d V_2.$$

So in the limit, and with mean zero for the X_i , there is no long-lasting difference between $(1/n) \sum_{i=1}^n (X_i - \bar{X}_n)^2$ and $(1/n) \sum_{i=1}^n X_i^2$. Make this statement more precise, in the vocabulary of chi-squared distributions.

(d) Use algebra to show that

$$A_{n,3} = (1/n) \sum_{i=1}^n (X_i - \bar{X}_n)^3 = M_{n,3} - 3M_{n,2}\bar{X}_n + 2\bar{X}_n^3.$$

Derive from this that

$$\sqrt{n}A_{n,3} \rightarrow_d V_3 - 3V_1,$$

and show that this is a $N(0, 6)$. For

$$S_n = (1/n) \sum_{i=1}^n \left(\frac{X_i - \bar{X}_n}{\hat{\sigma}} \right)^3 = \frac{A_{n,3}}{\hat{\sigma}^3},$$

called the sample skewness, show that $\sqrt{n}S_n \rightarrow_d N(0, 6)$, for normal data.

(e) We go on to the sample kurtosis

$$K_n = (1/n) \sum_{i=1}^n \left(\frac{X_i - \bar{X}_n}{\hat{\sigma}} \right)^4 - 3 = \frac{A_{n,4}}{\hat{\sigma}^4} - 3,$$

with $A_{n,4} = (1/n) \sum_{i=1}^n (X_i - \bar{X}_n)^4$. Establish that

$$\begin{pmatrix} \sqrt{n}(A_{n,4} - 3) \\ \sqrt{n}(A_{n,2} - 1) \end{pmatrix} \rightarrow_d \begin{pmatrix} V_4 \\ V_2 \end{pmatrix}.$$

Use the delta method to show $\sqrt{n}K_n \rightarrow_d V_4 - 6V_2 \sim N(0, 24)$.

(f) So now we know how different from zero the sample skewness and sample kurtosis can be expected to be (at least when n is large). Find the limit distribution of

$$Z_n = \frac{nS_n^2}{6} + \frac{nK_n^2}{24},$$

under the normality condition. Explain that the normality test, which rejects normality provided $Z_n > 5.991$, is an approximate 0.05 level test.

- (g) So we have constructed a perhaps useful test for normality, using the Z_n . For what type of alternatives to normality can this test be expected to be good? Find a non-normal distribution where the detection power of the test is not at all good.

3. Skating & running: sometimes apples $\approx$ bananas

MYSTERIOUSLY, the Olympic and World Championship level results for 800 m running and for 1500 m speedskating are pretty close. Seen from the appropriate statistical distance, newspaper items for one sport resemble those for the other. Check the Appendix for the list of the world's best results, for each year 2001 to 2025, for these two events.

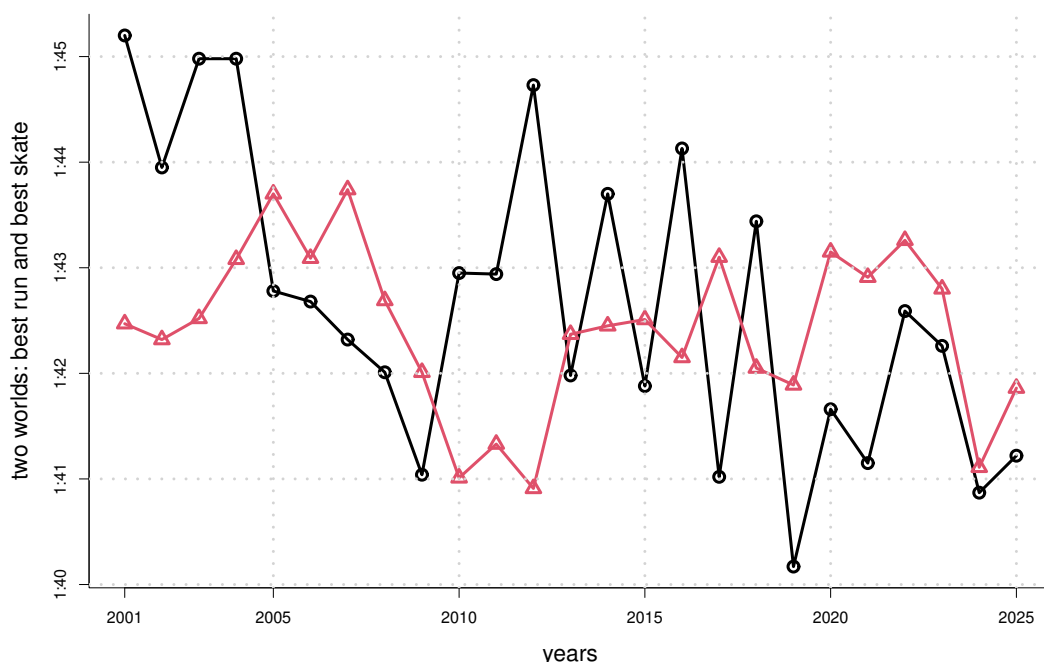


Figure A: the best results per season, for years 2001 to 2025, for 1500 m speedskating (black, small circles) and 800 m running (red, small triangles).

- (a) Get hold of the data, from the `twoworlds-data` file, translating results to seconds. Construct a version of Figure A. People from the relevant educated circles can rattle off the names, winning times, and records from that figure. We view the two-sample data as running X_i and skating Y_j for the $n = 25$ years coming from normal distributions (ξ_1, σ_1) and (ξ_2, σ_2) . Compute the empirical means (102.419 and 102.665) and standard deviations (0.780 and 1.429).
- (b) Construct 95 percent confidence intervals for σ_1 and σ_2 . The point estimate for $\rho = \sigma_2/\sigma_1$ is 1.833; find a confidence interval also for this ratio. Do this in as many as two ways: (i) use the exact F distribution of a ratio of chi-squareds; (ii) use the delta method based on the fact that $\sqrt{n}(\hat{\sigma} - \sigma) \rightarrow_d N(0, \frac{1}{2}\sigma^2)$ for a normal sample. Test the hypothesis $H_0: \sigma_1 = \sigma_2$, and comment on what you find.

- (c) Give the distributions of $t_1 = \sqrt{n}(\bar{X} - \xi_1)/\hat{\sigma}_1$ and $t_2 = \sqrt{n}(\bar{Y} - \xi_2)/\hat{\sigma}_2$, and show that these tend to two independent standard normals. Then show that

$$t = \frac{\bar{Y} - \bar{X} - (\xi_2 - \xi_1)}{(\hat{\sigma}_1^2/n + \hat{\sigma}_2^2/n)^{1/2}}$$

tends to the standard normal, for growing n . Set up a clear test for the null hypothesis $H_0: \xi_1 = \xi_2$, carry out the test, and comment. Construct also an approximate 95 percent confidence interval for $d = \xi_2 - \xi_1$, and explain why the word ‘approximate’ needs to be used for this particular task.

- (d) Discuss the modelling assumptions used here; are they plausible? For the X_i and Y_j data, compute their skewnesses and kurtoses, and from these the statistic

$$Z_n^* = \frac{nS_{n,x}^2}{6} + \frac{nK_{n,x}^2}{24} + \frac{nS_{n,y}^2}{6} + \frac{nK_{n,y}^2}{24}.$$

What is the limiting distribution for this test statistic, when n grows big, under normality conditions for the two datasets? Comment on what you find here.

- (e) Which of these fifty achievements do you value, or admire, the most?

Appendix: the dataset, and a few R details

The `twoworlds-data` can be read into your computer as follows:

```
skatingrunning <- matrix(scan("twoworlds-data",skip=8),byrow=T,ncol=6)
```

followed by `skating <- 60*skatingrunning[,2] + skatingrunning[,3]`, then perhaps `plot(year, skating)`, etc.

2025	1:41.86	# Wanyonyi	1:41.22	# Stolz, Calgary
2024	1:41.11	# Wanyonyi	1:40.87	# Stolz, SLC
2023	1:42.80	# Wanyonyi	1:42.26	# Stolz, SLC
2022	1:43.26	# Korir	1:42.59	# Nuis, Calgary
2021	1:42.91	# Amos	1:41.15	# Mantia, SLC
2020	1:43.15	# Brazier	1:41.66	# Nuis, SLC
2019	1:41.89	# Nijel Amos	1:40.17	# Nuis, SLC, WR
2018	1:42.05	# Korir	1:43.44	# DeHaitre, Calgary
2017	1:43.10	# Korir	1:41.02	# Yuskov, SLC, WR
2016	1:42.15	# Rudisha	1:44.13	# Yuskov, Kolomna
2015	1:42.51	# Amel Tuka	1:41.88	# Yuskov, Calgary
2014	1:42.45	# Nijel Amos	1:43.70	# Davis, SLC
2013	1:42.37	# Aman	1:41.98	# Davis, SLC
2012	1:40.91	# Rudisha, WR	1:44.73	# Morrison
2011	1:41.33	# Rudisha	1:42.94	# Skobrev, Calgary
2010	1:41.01	# Rudisha, WR	1:42.95	# Davis, SLC
2009	1:42.01	# Kipketer	1:41.04	# Davis, SLC, WR
2008	1:42.69	# Kipketer	1:42.01	# Morrison, Calgary, WR
2007	1:43.74	# Kipketer	1:42.32	# Davis, Calgary, WR
2006	1:43.09	# Kipketer	1:42.68	# Davis, Calgary, WR
2005	1:43.70	# Kipketer	1:42.78	# Hedrick, SLC
2004	1:43.08	# Kipketer	1:44.98	# Groothuis, Calgary
2003	1:42.52	# Kipketer	1:44.98	# Cheek, SLC
2002	1:42.32	# Kipketer	1:43.95	# Parra, SLC, WR
2001	1:42.47	# Kipketer	1:45.20	# Lee, Calgary, WR

This little dataset has been extracted from various background files, curated by Nils Lid Hjort. If sufficiently curious, you may check

Hjort, N.L. *Skøyteløpenes uutholdelige letthet*. Kronikk, Klassekampen, 15-xii-2025.

Hjort, N.L. *Ein 5000-meter frå framtida: 5.51,38*. Kronikk, Dag og Tid, 13-iii-2026.

Um Wiklunds tyrannosaurus rex-armrørsler i svingane, og meir frå nærsfylde umråde.