UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: STK4011/STK9011 — Statistical inference theory

Day of examination: Tuesday 8 December 2015.

Examination hours: 09.00 – 13.00.

This problem set consists of 3 pages.

Appendices: None.

Permitted aids: "List of formulas for STK1100 and STK1110" and "Summary of results for STK4011/9011".

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

Let $X_1, \ldots, X_n$ be independent Bernoulli variables with success probability p. So for each $i = 1, \ldots, n$, we have $P(X_i = 1) = 1 - P(X_i = 0) = p$.

a) Show that $T = \sum_{i=1}^{n} X_i$ is a sufficient statistic and that the family of pmfs of T has a monotone likelihood ratio.

b) Find the uniform most powerful test of $H_0 : p \leq p_0$ versus $H_1 : p > p_0$ and describe how you may determine the size of the test.

c) Derive the likelihood ratio test of $H_0 : p = p_0$ versus $H_1 : p \neq p_0$ and describe how you may use the test to find an approximate $1 - \alpha$ confidence interval for p.

Problem 2

We assume that $X \sim \text{gamma}(\alpha, \beta)$. So X has pdf

$$f(x|\alpha, \beta) = \begin{cases} \frac{1}{\beta^\alpha \Gamma(\alpha)} \, x^{\alpha-1} \, e^{-x/\beta} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

a) Show that

$$\mathrm{E}\left(X^\nu\right) = \beta^\nu \, \frac{\Gamma(\alpha + \nu)}{\Gamma(\alpha)}.$$

For which values of ν is the formula valid?

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We further assume that $Y \sim \text{gamma}(\gamma, \beta)$, and that X and Y are independent.
Let $U = X + Y$ and $V = X/(X + Y)$.

b) Derive the joint pdf of (U, V).

c) Show that $V \sim \text{beta}(\alpha, \gamma)$.

Problem 3

Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed with pdf

$$f(x|\theta) = \begin{cases} \frac{\theta^3}{2} x^2 e^{-\theta x} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

a) Find a complete sufficient statistic for θ.

b) Find the maximum likelihood estimator $\hat{\theta}$ of θ.

c) Show that

$$\mathrm{E}_\theta\, \hat{\theta} = \frac{3n}{3n-1}\, \theta$$

and

$$\mathrm{Var}_\theta\, \hat{\theta} = \frac{9n^2}{(3n-1)^2(3n-2)}\, \theta^2.$$

d) Find the best unbiased estimator of θ and show that its variance is larger than the Cramér-Rao lower bound.

e) Is there a function of θ, say $\tau(\theta)$, for which there exists an unbiased estimator whose variance attains the Cramér-Rao lower bound? If so, find it. If not, show why not.

Problem 4

Let $X_1, \ldots, X_n$ be independent and identically distributed with pdf

$$f_X(x|\theta) = \begin{cases} \frac{\theta^3}{2} x^2 e^{-\theta x} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

and let $Y_1, \ldots, Y_n$ be independent and identically distributed with pdf

$$f_Y(y|\phi) = \begin{cases} \frac{\phi^3}{2} y^2 e^{-\phi x} & \text{if } y > 0, \\ 0 & \text{if } y \leq 0. \end{cases}$$

Moreover, we assume that the X_i's and Y_i's are independent.

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a) Find the likelihood ratio test statistic for testing of $H_0 : \theta = \phi$ versus $H_1 : \theta \neq \phi$.

We introduce

$$T = \frac{\sum_{i=1}^{n} X_i}{\sum_{i=1}^{n} X_i + \sum_{j=1}^{n} Y_j}$$

b) Show that the likelihood ratio test rejects H_0 if $T(1-T) \leq k$ for a constant k.

c) Explain that an alternative formulation of the likelihood ratio test in question b is to reject H_0 if $T \leq a$ or $T \geq 1-a$ for a constant a. Find the distribution of T when H_0 is true and explain how you may use this distribution to find the value of a that gives a test with size α.

END