UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: STK4011/STK9011 — Statistical inference theory

Day of examination: Monday 5 December 2016.

Examination hours: 14.30 – 18.30.

This problem set consists of 3 pages.

Appendices: None.

Permitted aids: "List of formulas for STK1100 and STK1110" and "Summary of results for STK4011/9011".

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

Assume that $X_1, X_2, \ldots, X_n$ are independent and Poisson distributed with pmf

$$f(x|\theta) = \frac{\theta^x}{x!}e^{-\theta} \qquad \text{for } x = 0, 1, 2, \ldots,$$

where $\theta > 0$.

a) Show that $T = \sum_{i=1}^n X_i$ is a sufficient statistic.

b) Determine the distribution of T, and show that the family of pdfs of T has an increasing likelihood ratio.

c) Find the uniform most powerful test of $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$. Describe how you may determine the size of the test.

d) Derive the likelihood ratio test of $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$, and describe how you may use the test to find an approximate $1 - \alpha$ confidence interval for θ.

Problem 2

In this problem you may use that if $U \sim \text{gamma}(\alpha, \beta)$, then

$$\mathrm{E}\left(U^k\right) = \beta^k \frac{\Gamma(\alpha + k)}{\Gamma(\alpha)} \quad \text{for } k > -\alpha.$$

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Assume that $X_1, X_2, \ldots, X_n$ are independent and identically distributed with pdf

$$f(x|\theta) = \theta^2 x\, e^{-\theta x} \quad \text{if} \quad x > 0,$$

where $\theta > 0$.

a) Show that $T = \sum_{i=1}^n X_i$ is a complete and sufficient statistic.

b) Determine the distribution of T, and show that $\mathrm{E}(T^{-1}) = \theta/(2n-1)$ and $\mathrm{Var}(T^{-1}) = \theta^2/[(2n-1)^2(2n-2)]$.

c) Find the best unbiased estimator of θ and derive its variance.

d) Does the estimator in c) attain the Cramer-Rao lower bound?

Problem 3

Assume that $X_1, X_2, \ldots$ are independent and gamma$(\alpha, 1)$-distributed. Thus their pdf takes the form

$$f(x|\alpha) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x} \quad \text{if} \quad x > 0,$$

where $\alpha > 0$. It is well known that $\mathrm{E}X_i = \alpha$ and $\mathrm{Var}X_i = \alpha$ (and you shall not prove this).

a) Let α_n^* be the method of moments estimator of α based on $X_1, X_2, \ldots, X_n$. Derive an expression for α_n^* and explain why $\sqrt{n}(\alpha_n^* - \alpha) \to n(0, \alpha)$ in distribution.

b) Let $\widehat{\alpha}_n$ be the maximum likelihood estimator of α based on $X_1, X_2, \ldots, X_n$. We are not able to find an expression for $\widehat{\alpha}_n$, but it may be given as the solution of a non-linear equation. Find this equation using the digamma function $\psi(z) = \frac{d}{dz} \log \Gamma(z)$. (One may show that the digamma function is strictly increasing.)

c) Show that $\sqrt{n}(\widehat{\alpha}_n - \alpha) \to n(0, \sigma^2)$ in distribution, and find an expression for the asymptotic variance σ^2. You may here make use of the trigamma function $\psi_1(z) = \frac{d^2}{dz^2} \log \Gamma(z)$.

d) Find the asymptotic relative efficiency of the method of moments estimator α_n^* with respect to the maximum likelihood estimator $\widehat{\alpha}_n$. Use the table below to describe how the asymptotic relative efficiency depends on α and discuss what it tells you about the properties of the two estimators.

z	1	2	3	4	5	6	7	8	9	10
$\psi_1(z)$	1.645	0.645	0.395	0.284	0.221	0.181	0.154	0.133	0.118	0.105
$z\,\psi_1(z)$	1.645	1.290	1.185	1.135	1.107	1.088	1.075	1.065	1.058	1.052
$1/[z\,\psi_1(z)]$	0.608	0.775	0.844	0.881	0.904	0.919	0.930	0.939	0.946	0.951

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Problem 4

Assume that $X_1, X_2, \ldots, X_n$ are independent and uniformly distributed with pdf

$$f(x|\alpha,\beta) = \begin{cases} \frac{1}{\beta-\alpha} & \text{if} \quad \alpha \leq x \leq \beta, \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha < \beta$.

a) Show that $T(\mathbf{X}) = (U, V)$, where $U = X_{(1)} = \min X_i$ and $V = X_{(n)} = \max X_i$, is a sufficient statistic for (α, β).

b) Show that the joint pdf of (U, V) is given by

$$f_{UV}(u, v|\alpha, \beta) = \begin{cases} \frac{n(n-1)}{(\beta-\alpha)^n}(v-u)^{n-2} & \text{if} \quad \alpha \leq u < v \leq \beta, \\ 0 & \text{otherwise.} \end{cases}$$

c) Determine the joint pdf of (R, W), where $R = V - U$ is the range and $W = U$, and use this to find the marginal pdf of R.

d) Find an unbiased estimator of $\theta = \beta - \alpha$ and determine its variance.

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