UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: STK4011/STK9011 — Statistical inference theory

Day of examination: Friday December 15 2017

Examination hours: 2.30 pm – 6.30 pm.

This problem set consists of 2 pages.

Appendices: None

Permitted aids: Approved calculator, "List of formulas for STK1100 and STK1110 " and "Summary of results for STK4011/9011".

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

Assume that the observations $X_1, \ldots, X_n$ are independent and identically distributed from a Poisson distribution with parameter λ. The probability mass function, pmf, is then

$$f_X(x|\lambda) = \frac{\lambda^x}{x!}e^{-\lambda},\ x = 0, 1, \ldots$$

a) Show that $\hat{\lambda} = \frac{1}{n}\Sigma_{i=1}^n X_i$ is the maximum likelihood estimator for λ.

b) Explain why $\hat{\lambda}$ is consistent and find the asymptotic distribution when $n \to \infty$.

c) Use the delta-method to find the asymptotic distribution of $\sqrt{\hat{\lambda}}$.

d) Consider the hypothesis test $H_0 : \lambda \leq \lambda_0$ versus $H_1 : \lambda > \lambda_0$.

   Find a test with approximate size α for this problem based on $\sqrt{\hat{\lambda}}$ using the results from part c). What is the approximate power function?

Problem 2

Assume that the observations $X_1, \ldots, X_n$ are independent and identically distributed Bernoulli variables with success parameter p so the probability mass function, pmf, is

$$P(X = x) = p^x(1 - p)^{1-x},\ x = 0, 1$$

Also, assume in the following that $n \geq 3$.

a) What is the requirement which a statistic must satisfy in order to be sufficient for p? Show using the definition, i.e. without using that the Bernoulli distributions belong to the exponential family of distributions, that $\Sigma_{i=1}^n X_i$ satisfies this requirement. Is it minimal sufficient?

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b) What does it mean that a statistic is complete? Show, also from the definition without using that the Bernoulli distributions belong to the exponential family of distributions, that $\Sigma_{i=1}^n X_i$ is complete.

c) Explain why $I(X_1 = 1, X_2 = 1)$ is an unbiased estimator for p^2. Here $I(\cdot)$ is the indicator function.

d) Use the Rao-Blackwell recipe to construct an unbiased estimator of p^2 based on $\Sigma_{i=1}^n X_i$ and explain why this estimator is best unbiased or uniformly minimum variance unbiased estimator UMVUE.

e) Find the lower bound for the variance of the estimator from part d) from the the Cramér-Rao inequality.

Problem 3

Assume that the observations $X_1, \ldots, X_n$ are independent and identically distributed variables with probability density function, pdf, defined as

$$f_X(x|\theta) = \begin{cases} \exp(-(x-\theta)) & \theta \leq x, -\infty < \theta < \infty \\ 0 & \text{else} \end{cases}$$

where θ is an unknown parameter.

a) Explain why this class of distributions does not belong to the exponential family of distributions.

b) Find the maximum likelihood estimator, $\hat{\theta}$, for θ. Find its distribution and explain why it is sufficient.

c) Find the likelihood ratio test, LRT, for the testing problem

$$H_0 : \theta \leq \theta_0 \text{ versus } H_1 : \theta > \theta_0.$$

Determine the rejection region for the LRT having size α.

Problem 4

The measurements $X_1, \ldots, X_n$ from an instrument for measuring a known standard are assumed to be a random sample from a $n(\mu_0, \sigma^2)$ distribution with known mean μ_0 and unknown variance σ^2.

a) Find the uniform maximum power, UMP, test with level α for testing the hypothesis

$$H_0 : \sigma = \sigma_0 \text{ versus } H_1 : \sigma > \sigma_0.$$

b) What is the power function for the test in part a)?

c) Show that the test from part a) also is UMP with level α for testing

$$H_0 : \sigma \leq \sigma_0 \text{ versus } H_1 : \sigma > \sigma_0.$$

END