UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: IN ??? — ???

Day of examination: ??. ??. ????

Examination hours: ??.??–??.??

This problem set consists of 0 pages.

Appendices: None

Permitted aids: Any

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

Assume that the observations $X_1, \ldots, X_n$ are independent, identically distributed from a Poisson distribution with parameter λ. The probability mass function, pmf, is then

$$f_X(x|\lambda) = \frac{\lambda^x}{x!}e^{-\lambda},\ x = 0, 1, \ldots$$

a) Show that $\hat{\lambda} = \frac{1}{n}\Sigma_{i=1}^n X_i$ is the maximum likelihood estimator for λ.

b) Explain why $\hat{\lambda}$ is consistent and find the asymptotic distribution when $n \to \infty$.

c) Use the delta-method to find the asymptotic distribution of $\sqrt{\hat{\lambda}}$.

d) Consider the hypothesis test $H_0 : \lambda \leq \lambda_0$ versus $H_1 : \lambda > \lambda_0$.

   Find a test with approximate size α for this problem based on $\sqrt{\hat{\lambda}}$ using the results from part c). What is the approximate power function?

Problem 2

Assume that the observations $X_1, \ldots, X_n$ are independent, identically distributed Bernoulli variables with success parameter p so the probability mass function, pmf, is

$$P(X = x) = p^x(1-p)^{1-x},\ x = 0, 1$$

a) What is the requirement which a statistic must satisfy in order to be sufficient? Show, without using that the Bernoulli distributions belong to the exponential family of distributions, that $\Sigma_{i=1}^n X_i$ satisfies this requirement. Is it minimal sufficient?

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b) What does it mean that a statistic is complete? Show, also without using that the Bernoulli distributions belong to the exponential family of distributions, that $\Sigma_{i=1}^n X_i$ is complete.

c) Explain why $I(X_1 = 1, X_2 = 1)$ is an unbiased estimator for p^2. Here $I(\cdot)$ is the indicator function.

d) Use the Rao-Blackwell recipe to construct an unbiased estimator of p^2 based on $\Sigma_{i=1}^n X_i$ and explain why this estimator is best unbiased or uniformly minimum variance unbiased estimator UMVUE.

e) Is the variance of the estimator from part d) equal to the lower bound in the Cramer-Rao inequality?

Problem 3

Assume that the observations $X_1, \dots, X_n$ are independent, identically distributed variables with probability density function, pdf, is

$$f_X(x|\theta) = \begin{cases} \exp(-(x-\theta)) & \theta < x, -\infty < \theta < \infty \\ 0 & \text{else} \end{cases}$$

a) Explain why this class of distributions does not belong to the exponential family of distributions.

b) Find the maximum likelihood estimator, $\hat{\theta}$, for θ. Find its distribution and explain why it is sufficient.

c) Find the likelihood ratio test, LRT for the testing problem

$$H_0 : \theta \leq \theta_0 \text{ versus } H_1 : \theta > \theta_0.$$

d) Find the uniform maximum power, UMP, test with level α for the testing the hypothesis

$$H_0' : \theta = \theta_0 \text{ versus } H_1' : \theta = \theta_1$$

where $\theta_1 > \theta_0$.

e) Show that the test from part d) also is UMP with level α for testing

$$H_0' : \theta = \theta_0 \text{ versus } H_1 : \theta > \theta_0.$$

f) Show that the test from part d) also is UMP with level α for testing

$$H_0 : \theta \leq \theta_0 \text{ versus } H_1 : \theta > \theta_0.$$

END

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