UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: STK4011/STK9011 — Statistical inference theory

Day of examination: Friday December 7th 2018

Examination hours: 9.00 am – 1.00 pm.

This problem set consists of 2 pages.

Appendices: None

Permitted aids: Approved calculator, "List of formulas for STK1100 and STK1110 " and "Summary of results for STK4011/9011".

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All parts of the problems count equally to the final grade.

Problem 1

Assume that the observations $X_1, \dots, X_n$ where $n \geq 3$, are independent and identically Bernoulli (p) distributed. The probability mass function, pmf, is then

$$f_X(x|p) = p^x(1-p)^{1-x},\ x = 0, 1.$$

a) Explain why the Bernoulli(p) distribution belongs to the exponential family of distributions. Find a minimal sufficient statistic for p. Is it complete?

b) Find the maximum likelihood estimator, MLE, for p and describe its asymptotic distribution.

c) Consider $\tau(p) = p(1-p)$. What is the maximum likelihood estimator for $\tau(p)$? Find the asymptotic distribution both when $p \neq \frac{1}{2}$ and $p = \frac{1}{2}$

d) Determine the best unbiased estimator, BUE, for $\tau(p)$.

e) Consider the case where $p \neq \frac{1}{2}$. Find the asymptotic distribution of the best unbiased estimator from part d) in this case. What is the relative asymptotic efficiency of the best unbiased estimator with respect to the maximum likelihood estimator for $\tau(p)$ in the situation where $p \neq \frac{1}{2}$?

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Problem 2

Assume the variables $X_1, \dots, X_n$ are a random sample from a $n(\mu, 1)$ distribution with unknown expectation μ.

a) Find a minimal sufficient statistic for μ.

b) What is the Cramer-Rao lower bound for $\tau(\mu) = \mu^2$?

c) Why is $\bar{X}^2 - \frac{1}{n}$, where $\bar{X} = \sum_{i=1}^{n} X_i$, the best unbiased estimator, BUE, for $\tau(\mu) = \mu^2$.

Problem 3

Assume that the random variables $X_1, \dots, X_n$ are independent, identically distributed with $X_i \sim n(0, \sigma^2)$ where $\sigma > 0$ is the unknown parameter.

a) Consider testing the null hypothesis $H_0' : \sigma = \sigma_0$ versus the alterternative $H_1' : \sigma = \sigma_1$ where $\sigma_1 > \sigma_0$. Let $\mathcal{R}$ be the rejection region. Denote the test function ϕ as $\phi = I_{\mathcal{R}}$ where $I_{\mathcal{R}}$ is the indicator function of $\mathcal{R}$, i.e. $I_{\mathcal{R}}(x) = 1$ if $x \in \mathcal{R}$ and $I_{\mathcal{R}}(x) = 0$ if $x \notin \mathcal{R}$.

   Explain why the uniformly most poweful , UMP, test has the form

$$\phi = \begin{cases} 1 & \text{if} \quad \sum_{i=1}^{n} X_i^2 > c \\ 0 & \text{if} \quad \sum_{i=1}^{n} X_i^2 < c. \end{cases}$$

b) Explain how c can be chosen so that the test has size α?

c) Now, consider the more general setup, testing the null hypothesis $H_0 : \sigma \leq \sigma_0$ versus the alternative $H_1 : \sigma > \sigma_0$. Explain why the test from part a) is uniformly most powerful also in this situation.

d) What is the power function of the test from part c)? Is the test unbiased?

Problem 4

Assume the variables $X_1, \dots, X_n$ are a random sample from a $n(\mu, 1)$ distribution with unknown expectation μ.

a) What is meant by an ancillary statistic in this situation. State Basu's theorem.

b) Show that $X_1 - \bar{X}$ is an ancillary statistic.

c) Explain why $X_1 - \bar{X}$ and $\bar{X}$ are independent.

END