UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in:	STK4011/9011 — Statistical Inference Theory
Day of examination:	Friday, 4 December 2020
Examination hours:	9.00 – 13.00

This problem set consists of 2 pages.

Appendices:	None
Permitted aids:	All examination aids are allowed. However, it is not allowed to cooperate or communicate with others regarding the exam.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

Assume that $X_1, \ldots, X_n$ is a random sample from a population with probability density function

$$f(x \mid \theta) = \frac{\theta}{x^{\theta+1}}, \quad 1 < x < \infty,$$

where $\theta > 1$.

(a) Show that the method of moments estimator of θ is

$$\hat{\theta}_{MM} = \frac{\overline{X}}{\overline{X} - 1}.$$

(b) Show that the method of moments estimator is consistent.

(c) Show that the maximum likelihood estimator of θ is

$$\hat{\theta}_{ML} = \frac{n}{T}, \quad \text{where } T = \sum_{i=1}^{n} \log(X_i).$$

(d) It can be shown that $T \sim \text{Gamma}(n, 1/\theta)$, that is,

$$f(t \mid n, \theta) = \frac{\theta^n}{\Gamma(n)} t^{n-1} e^{-t\theta}, \quad 0 \le t < \infty.$$

Use this fact to show that

$$E(\hat{\theta}_{ML}) = \frac{n\theta}{n-1},$$

and explain what it implies in terms of the properties of $\hat{\theta}_{ML}$.

(e) Show that $T = \sum_{i=1}^{n} \log(X_i)$ is a complete sufficient statistic for θ.

(f) Find the best unbiased estimator of θ.

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Problem 2

Assume that $X_1, \ldots, X_n$ is a random sample from a population with probability mass function

$$f(x \mid \theta) = \theta(1-\theta)^{x-1}, \quad x = 1, 2, 3, \ldots,$$

where $\theta \in (0, 1)$.

(a) Show that the moment generating function of a variable X with the above probability mass function is

$$M_X(t) = \frac{\theta e^t}{1-(1-\theta)e^t}, \quad \text{for } t < -\log(1-\theta). \tag{1}$$

(b) Use the moment generating function (1) to show that

$$E(X) = \frac{1}{\theta} \quad \text{and} \quad Var(X) = \frac{1-\theta}{\theta^2}.$$

(c) Show that $T = \sum_{i=1}^n X_i$ is a minimal sufficient statistic for θ.

(d) It can be shown (you don't have to show it) that the maximum likelihood estimator of θ is

$$\hat{\theta}_{ML} = \frac{1}{\overline{X}}, \quad \text{where } \overline{X} = \frac{1}{n}\sum_{i=1}^n X_i.$$

Show that the asymptotic variance of $\hat{\theta}_{ML}$ is $\theta^2(1-\theta)$.

(e) Find the maximum likelihood estimator of $\mu(\theta) = E(X)$ and show that it attains the Cramer-Rao Lower Bound.

Problem 3

Assume that $X_1, \ldots, X_n$ is a random sample from a population with probability density function

$$f(x \mid \theta) = \theta e^{-\theta x}, \; x \geq 0,$$

where $\theta > 0$. We are interested in testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$.

(a) Derive the likelihood ratio test statistic and use it to construct a test that is approximately size α.

(b) Show that the likelihood ratio test is equivalent to rejecting H_0 if

$$\overline{X} \leq c_1 \text{ or } \overline{X} \geq c_2$$

for some constants c_1 and c_2.

(c) Explain how to select the constants c_1 and c_2 in (b) in order to obtain a test of size α.

THE END - GOOD LUCK!