Hierarchical model

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(a). Set up a model for the data in Table 3.3 so that, for $j = 1, \ldots, 10$, the observed number of bicycles at location $j$ is binomial with unknown probability $\theta_j$ and sample size equal to the total number of vehicles (bicycles included) in that block. The parameter $\theta_j$ can be interpreted as the underlying or 'true' proportion of traffic at location $j$ that is bicycles. Assign a beta population distribution for the parameters $\theta_j$ and a non-informative hyper-prior distribution. Write down the joint posterior distribution i.e $p(\theta, \alpha, \beta | y)$. 
Joint Posterior Distribution

\[ p(\theta, \alpha, \beta|y) \propto p(y|\theta, \alpha, \beta) \cdot p(\theta|\alpha, \beta)p(\alpha, \beta) \]

Likelihood \hspace{2cm} joint prior distribution \( p(\theta, \alpha, \beta) \)
Let us consider $y_1, y_2, \ldots, y_{10}$ are the number of bicycles at ten different locations, and $n_1, n_2, \ldots, n_{10}$ are the total vehicles including bicycles at location 1 to 10, According to the statement, $y_j$ can be considered to be generated from a binomial process with known $\theta_j$;

$$y_j|\theta_j \sim Bin(n_j, \theta_j), j = 1, 2, \ldots, 10$$
Joint Posterior Distribution

\[
p(\theta, \alpha, \beta | y) \propto p(y | \theta, \alpha, \beta) \cdot p(\theta | \alpha, \beta) \cdot p(\alpha, \beta)
\]

Likelihood \hspace{1cm} joint prior distribution \( p(\theta, \alpha, \beta) \)

Let us consider \( y_1, y_2, \ldots, y_{10} \) are the number of bicycles at ten different locations, and \( n_1, n_2, \ldots, n_{10} \) are the total vehicles including bicycles at location 1 – 10, According to the statement, \( y_j \) can be considered to be generated from a binomial process with known \( \theta_j \);

\[
y_j | \theta_j \sim \text{Bin}(n_j, \theta_j), j = 1, 2, \ldots, 10
\]

\[
p(y | \theta, \alpha, \beta) = \prod_{j=1}^{10} \binom{n_j}{y_j} \theta_j^{y_j} (1 - \theta_j)^{n_j - y_j}
\]
Joint Posterior Distribution

\[ p(\theta, \alpha, \beta | y) \propto p(y | \theta, \alpha, \beta) \cdot p(\theta | \alpha, \beta) p(\alpha, \beta) \]

Likelihood \hspace{1cm} joint prior distribution \( p(\theta, \alpha, \beta) \)

Let us consider \( y_1, y_2, \ldots, y_{10} \) are the number of bicycles at ten different locations, and \( n_1, n_2, \ldots, n_{10} \) are the total vehicles including bicycles at location 1 – 10, According to the statement, \( y_j \) can be considered to be generated from a binomial process with known \( \theta_j \);

\[ y_j | \theta_j \sim Bin(n_j, \theta_j), j = 1, 2, \ldots, 10 \]

\[ p(y | \theta, \alpha, \beta) = \prod_{j=1}^{10} \binom{n_j}{y_j} \theta_j^{y_j} (1 - \theta_j)^{n_j - y_j} \]

Similarly, \( \theta_j \) has \textit{Beta} distribution with hyper-parameter \( \alpha \) and \( \beta \),

\[ \theta_j \sim \text{Beta}(\alpha, \beta) \]
Joint Posterior Distribution

\[ p(\theta|\alpha, \beta) = \prod_{j=1}^{10} \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta_j^{\alpha-1}(1 - \theta_j)^{\beta-1} \]

One of the possible non-informative prior distribution for the hyper parameters is;

\[ p(\alpha, \beta) \propto (\alpha + \beta)^{-5/2} \]
Joint Posterior Distribution

\[ p(\theta | \alpha, \beta) = \prod_{j=1}^{10} \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta_j^{\alpha-1} (1 - \theta_j)^{\beta-1} \]

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By putting things together,

\[ p(\theta, \alpha, \beta | y) \propto p(\alpha, \beta) \prod_{j=1}^{10} \theta_j^{y_j} (1 - \theta_j)^{n_j-y_j} \prod_{j=1}^{10} \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta_j^{\alpha-1} (1 - \theta_j)^{\beta-1} \]
Joint Posterior Distribution

\[ p(\theta|\alpha, \beta) = \prod_{j=1}^{10} \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta_j^{\alpha - 1}(1 - \theta_j)^{\beta - 1} \]

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\[ p(\theta, \alpha, \beta|y) \propto (\alpha + \beta)^{-5/2} \left( \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \right)^{10} \prod_{j=1}^{10} \theta_j^{y_j}(1 - \theta_j)^{n_j-y_j} \theta_j^{\alpha - 1}(1 - \theta_j)^{\beta - 1} \]
Joint Posterior Distribution

\[ p(\theta, \alpha, \beta | y) \propto (\alpha + \beta)^{-5/2} \left( \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \right)^{10} \prod_{j=1}^{10} \theta_j^{\alpha+y_j-1}(1 - \theta_j)^{\beta+n_j-y_j-1} \]
(b). Compute the marginal posterior density of the hyperparameters and draw simulations from the joint posterior distribution of the parameters and hyperparameters.

The marginal distribution of the hyperparameters can be computed through the following algebraic expression,

\[ p(\alpha, \beta | y) = \frac{p(\theta, \alpha, \beta | y)}{p(\theta | \alpha, \beta, y)} \]
Marginal Distribution of the Hyperparameters

\[ p(\theta, \alpha, \beta | y) \propto (\alpha + \beta)^{-5/2} \left( \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \right)^{10} \prod_{j=1}^{10} \theta_j^{\alpha+y_j-1}(1 - \theta_j)^{\beta+n_j-y_j-1} \]

\[ p(\theta | \alpha, \beta, y) = \prod_{j=1}^{10} \frac{\Gamma(\alpha + \beta + n_j)}{\Gamma(\alpha + y_j)\Gamma(\beta + n_j - y_j)} \theta_j^{\alpha+y_j-1}(1 - \theta_j)^{\beta+n_j-y_j-1} \]

\[ p(\alpha, \beta | y) \propto (\alpha + \beta)^{-5/2} \left( \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \right)^{10} \prod_{j=1}^{10} \frac{\Gamma(\alpha + y_j)\Gamma(\beta + n_j - y_j)}{\Gamma(\alpha + \beta + n_j)} \]
Usually Gibbs sampling is used to simulate from a distribution. But it works only if we know the normalized form of the distribution. In this case $p(\alpha, \beta | y)$ is not available in normalized form. In order to sample from unnormalized distribution, different techniques like acceptance-rejection methods, Metropolis-Hastings algorithm can be used.

For the given values of $\alpha$ and $\beta$, we can simulate $p(\theta_j | \alpha, \beta, y)$ because its normalized form is available.
(c). Compare the posterior distributions of the parameters $\theta_j$ to the raw proportions. (number of bicycles / total number of vehicles) in location $j$. 
(d). Give a 95% posterior interval for the average underlying proportion of traffic that is bicycles.

\[
p < -\frac{\alpha}{\alpha + \beta}
\]

\[
quantile(p, c(0.025, 0.975))
\]

2.5%  97.5%
0.1465  0.2938

It shows that 14.6% to 29.4% of the vehicles are bicycles.
Predictive distribution

(e). A new city block is sampled at random and is a residential street with a bike route. In an hour of observation, 100 vehicles of all kinds go by. Give a 95% posterior interval for the number of those vehicles that are bicycles.

- Essentially, we are looking for posterior predictive distribution $p(\tilde{y}|y)$.
- In order to get a sample from it, we need to generate new values of $\tilde{\theta}$ from $\alpha, \beta$ samples.

$$\tilde{\theta} \sim Beta(\alpha, \beta)$$

- Now using these values, we can sample from binomial distribution with $n = 100$ i.e.

$$\tilde{y}|\tilde{\theta} \sim Bin(100, \tilde{\theta})$$
Predictive distribution

- \( \theta_{\text{new}} \sim -\text{rbeta}(S, \alpha, \beta) \)
- \( y_{\text{new}} \sim -\text{rbinom}(S, 100, \theta_{\text{new}}) \)
- \( \text{quantile}(y_{\text{new}}, c(0.025, 0.975)) \)
- 2.5% 97.5%
- 3 49

which means 3 to 49 bicycles will pass from the residential street with bike route (newly sampled) with the probability of 95%.
(f). Was the beta distribution for the $\theta_j$’s reasonable?

The plot of $\theta$ vs $\frac{y}{n}$ fits well which indicates the values of $\theta$ are reasonable.
Thanks