Solutions to selected exercises from BDA3

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Exercise 1.3

We have

$$\begin{aligned}\Pr(C = Xx|C = B, P1 = B, P2 = B) =& \frac{\Pr(C = Xx, C = B, P1 = B, P2 = B)}{\Pr(C = B, P1 = B, P2 = B)} \\ =& \frac{\Pr(C = Xx, P1 = B, P2 = B)}{\Pr(C = B, P1 = B, P2 = B)}\end{aligned}$$

due to that $C = Xx$ implies $C = B$ (brown eyes). Further,

$$\begin{aligned}\Pr(C =& Xx, P1 = B, P2 = B) \\ &= \Pr(C = Xx, P1 = Xx, P2 = Xx) + \Pr(C = Xx, P1 = Xx, P2 = XX) + \\ &\quad \Pr(C = Xx, P1 = XX, P2 = Xx) + \Pr(C = Xx, P1 = XX, P2 = XX) \\ &= \Pr(C = Xx|P1 = Xx, P2 = Xx) \Pr(P1 = Xx, P2 = Xx) + \\ &\quad \Pr(C = Xx|P1 = Xx, P2 = XX) \Pr(P1 = Xx, P2 = XX) + \\ &\quad \Pr(C = Xx|P1 = XX, P2 = Xx) \Pr(P1 = XX, P2 = Xx) + \\ &\quad \Pr(C = Xx|P1 = XX, P2 = XX) \Pr(P1 = XX, P2 = XX) \\ &= 0.5 * 2p(1-p)2p(1-p) + 0.5 * (1-p)^2 2p(1-p) + 0.5 * (1-p)^2 2p(1-p) + 0 \\ &= 2p^2(1-p)^2 + 2p(1-p)^3\end{aligned}$$

while

$$\Pr(C = B, P1 = B, P2 = B) = \Pr(C = Xx, P1 = B, P2 = B) + \Pr(C = XX, P1 = B, P2 = B)$$

where now

$$\begin{aligned}\Pr(C =& XX, P1 = B, P2 = B) \\ &= \Pr(C = XX, P1 = Xx, P2 = Xx) + \Pr(C = XX, P1 = Xx, P2 = XX) + \\ &\quad \Pr(C = XX, P1 = XX, P2 = Xx) + \Pr(C = XX, P1 = XX, P2 = XX) \\ &= \Pr(C = XX|P1 = Xx, P2 = Xx) \Pr(P1 = Xx, P2 = Xx) + \\ &\quad \Pr(C = XX|P1 = Xx, P2 = XX) \Pr(P1 = Xx, P2 = XX) + \\ &\quad \Pr(C = XX|P1 = XX, P2 = Xx) \Pr(P1 = XX, P2 = Xx) + \\ &\quad \Pr(C = XX|P1 = XX, P2 = XX) \Pr(P1 = XX, P2 = XX) \\ &= 0.25 * 2p(1-p)2p(1-p) + 0.5 * (1-p)^2 2p(1-p) + 0.5 * (1-p)^2 2p(1-p) + 1(1-p)^4 \\ &= 2p^2(1-p)^2 + 2p(1-p)^3 + (1-p)^4\end{aligned}$$

which combined gives

$$\Pr(C = Xx|C = B, P1 = B, P2 = B) = \frac{2p}{1+2p}$$

Exercise 1.7

Consider two possible outcomes: Y which corresponds to that the contestant choose the right box in the first place and N the complement. Then $\Pr(Y) = 1/3, \Pr(N) = 2/3$.

Now, if the outcome is N, then the price must be one of the not chosen boxed. Monty then has to choose the other box which means that the other box that the contestant did not choose must be the one with the price.

Therefore, if the contestant choose to switch, the probability for getting the price is $2/3$, so this is the best strategy!

Exercise 2.14

- We have

$$\begin{aligned}
p(\theta|y) &\propto \exp\left(-\tfrac{1}{2}\left(\tfrac{(y-\theta)^2}{\sigma^2} + \tfrac{(\theta-\mu_0)^2}{\tau_0^2}\right)\right) \\
&\propto \exp\left(-\tfrac{1}{2}\left[\tfrac{1}{\sigma^2}(y^2 - 2\theta y + \theta^2) + \tfrac{1}{\tau_0^2}(\theta^2 - 2\mu_0\theta + \mu_0^2)\right]\right) \\
&\propto \exp\left(-\tfrac{1}{2}\left[\left(\tfrac{1}{\sigma^2} + \tfrac{1}{\tau_0^2}\right)\theta^2 - 2\left(\tfrac{y}{\sigma^2} + \tfrac{\mu_0}{\tau_0^2}\right)\theta\right]\right) \\
&\propto \exp\left(-\tfrac{1}{2}\left(\tfrac{1}{\sigma^2} + \tfrac{1}{\tau_0^2}\right)\left(\theta^2 - 2\frac{\frac{y}{\sigma^2} + \frac{\mu_0}{\tau_0^2}}{\frac{1}{\sigma^2} + \frac{1}{\tau_0^2}}\theta\right)\right) \\
&= \exp\left(-\tfrac{1}{2\tau_1^2}\left(\theta^2 - 2\mu_1\theta\right)\right) \\
&\propto \exp\left(-\tfrac{1}{2\tau_1^2}\left(\theta^2 - 2\mu_1\theta + \mu_1^2\right)\right) \\
&= \exp\left(-\tfrac{1}{2\tau_1^2}(\theta - \mu_1)^2\right) \\
\propto N(\mu_1, \tau_1^2)
\end{aligned}$$

- Assume now a prior $N(\mu_1, \tau_1^2)$ and a new observation y_2. Then we can update to the posterior by replacing μ_0 with μ_1 and τ_0 with τ_1 in eq (2.10) and obtain

$$\begin{aligned}
\mu_2 =& \frac{\frac{1}{\tau_1}\mu_1 + \frac{1}{\sigma^2}y_2}{\frac{1}{\tau_1} + \frac{1}{\sigma^2}} = \frac{\frac{\mu_0}{\tau_0^2} + \frac{y_1}{\sigma^2} + \frac{1}{\sigma^2}y_2}{\frac{1}{\sigma^2} + \frac{1}{\tau_0^2 + \frac{1}{\sigma^2}}} = \frac{\frac{\mu_0}{\tau_0^2} + \frac{y_1 + y_2}{\sigma^2}}{\frac{2}{\sigma^2} + \frac{1}{\tau_0^2}} \\
\frac{1}{\tau_2} =& \frac{1}{\tau_1} + \frac{1}{\sigma^2} = \frac{1}{\tau_0} + \frac{2}{\sigma^2}
\end{aligned}$$

  and we can show the general formula by induction

In general we have for conditionally independent data

$$p(\theta|y_1, y_2) \propto p(\theta)p(y_1|\theta)p(y_2|\theta) \propto \qquad\qquad p(\theta|y_1)p(y_2|\theta)$$

and more generally

$$p(\theta|y_1, y_2, ..., y_n) \propto p(\theta|y_1, ..., y_{n-1})p(y_n|\theta)$$

showing that we can recursively update our posterior as new data arrive.

3.1

- Marginal distribution for $(\theta_1, ..., \theta_{k-2})$: We have

$$\begin{aligned}
p(\theta_1, ..., \theta_{k-2}) =& \int_{\theta_{k-1}} p(\theta_1, ..., \theta_{k-1}) d\theta_{k-1} \\
\propto& \prod_{j=1}^{k-2} \theta_j^{\alpha_j - 1} \int_{\theta_{k-1}} \theta_{k-1}^{\alpha_{k-1}-1} (1 - \theta_1 - \cdots - \theta_{k-2} - \theta_{k-1})^{\alpha_k - 1} d\theta_{k-1} \\
=& \prod_{j=1}^{k-2} \theta_j^{\alpha_j - 1} (1 - \theta_1 - \cdots - \theta_{k-2})^{\alpha_{k-2}+\alpha_{k-1}-2} \times \\
& \int_{\theta_{k-1}} \left(\frac{\theta_{k-1}}{1 - \theta_1 - \cdots - \theta_{k-2}} \right)^{\alpha_{k-1}-1} \frac{1 - \theta_1 - \cdots - \theta_{k-2} - \theta_{k-1}}{1 - \theta_1 - \cdots - \theta_{k-2}}^{\alpha_k - 1} d\theta_{k-1} \\
\propto& \prod_{j=1}^{k-2} \theta_j^{\alpha_j - 1} (1 - \theta_1 - \cdots - \theta_{k-2})^{\alpha_{k-2}+\alpha_{k-1}-2}
\end{aligned}$$

  where we have used that the integrand is proportional to a Beta density. The last expression is proportional to a Dirichlet distribution for $(\theta_1, ..., \theta_{k-2})$.

Exercise 3.7

- Poisson distributions:

$$\begin{aligned}
p(b, v|\theta_b, \theta_v) =& \frac{\theta_b^b \exp(-\theta_b)}{b!} \frac{\theta_v^v \exp(-\theta_v)}{v!} \\
\propto& \theta_b^b \theta_v^v \exp(-(\theta_b + \theta_v))
\end{aligned}$$

- Binomal distribution:

$$\begin{aligned}
p(b|b + v, \theta_b, \theta_v) =& \binom{b+v}{b} p^b (1-p)^v \\
\propto& \left(\frac{\theta_b}{\theta_b + \theta_v} \right)^b \left(\frac{\theta_v}{\theta_b + \theta_v} \right)^v \\
=& \theta_b^b \theta_v^v (\theta_b + \theta_v) * b + v \\
\propto& \theta_b^b \exp(-\theta_b) \theta_v^v \exp(-\theta_v)
\end{aligned}$$

  These likelihoods are different!

  In order to make them equal, we must for the binomial distribution take into account that also $b + v$ is stochastic. Assuing $b + v$ follows a Poisson distribution with intensity $\theta_b + \theta_v$, we get

$$\begin{aligned}
p(b, v|\theta_b, \theta_v) =& p(b + v|\theta_b, \theta_v) p(b|b + v, \theta_b, \theta_v) \\
\propto& \frac{(\theta_b + \theta_v)^{b+v}}{(b + v)! \exp(-(\theta_b + \theta_v)} \left(\frac{\theta_b}{\theta_b + \theta_v} \right)^b \left(\frac{\theta_v}{\theta_b + \theta_v} \right)^v \\
\propto& \theta_b^b \theta_v^v \exp(-(\theta_b + \theta_v)).
\end{aligned}$$

Exercise 3.13

We have that

$$p(\mu|y,\Sigma) \propto \exp\left(-\frac{1}{2}\underbrace{\left((\mu-\mu_0)^T\Lambda_0^{-1}(\mu-\mu_0)+\sum_{i=1}^{n}(y_i-\mu)^T\Sigma^{-1}(y_i-\mu)\right)}_{g(\mu)}\right)$$

Now

$$\begin{aligned}
g(\mu) =&\mu^T\Lambda_0^{-1}\mu - 2\mu_0\Lambda_0^{-1}\mu + n\mu^T\Sigma^{-1}\mu - 2\sum_{i=1}^{n} y_i\Sigma^{-1}\mu + C\\
=&\mu^T[\Lambda_0^{-1}+n\Sigma^{-1}]\mu - 2[\mu_0^T\Lambda_0^{-1}+\sum_{i=1}^{n} y_i^T\Sigma^{-1}]\mu + C\\
=&\mu^T[\Lambda_0^{-1}+n\Sigma^{-1}]\mu - 2[\mu_0^T\Lambda_0^{-1}+n\bar{y}^T\Sigma^{-1}]\mu + C\\
=&(\mu-[\Lambda_0^{-1}+n\Sigma^{-1}]^{-1}[\Lambda_0^{-1}\mu_0+n\Sigma^{-1}\bar{y}])^T[\Lambda_0^{-1}+n\Sigma^{-1}]\times\\
&(\mu-[\Lambda_0^{-1}+n\Sigma^{-1}]^{-1}[\Lambda_0^{-1}\mu_0+n\Sigma^{-1}\bar{y}]) + C'\\
=&(\mu-\mu_n)^T\Lambda_n^{-1}(\mu-\mu_n)
\end{aligned}$$

showing the required result.

Exercise 3.15

- Denote by y_{-t} all observations except t. We have (assuming $1 < t < T$)

$$\begin{aligned}
p(y_t|y_{-t}) \propto& p(y_t, y_{-t})\\
=&p(y_1)\prod_{s=2}^{T} p(y_s|y_{s-1})\\
\propto&p(y_t|y_{t-1})p(y_{t+1}|y_t) \qquad \text{ignoring all terms not depending on } y_t
\end{aligned}$$

  which shows that the distribution of y_t only depend on y_{t-1} and y_{t+1}.

- We have

$$\begin{aligned}
\log p(y_t|y_{t-1}, y_{t+1}) =& C + \log p(y_t|y_{t-1}) + \log p(y_{t+1}|y_t)\\
=&C - \frac{1}{2}(y_t - 0.8y_{t-1})^2 - \frac{1}{2}(y_{t+1} - 0.8y_t)^2\\
=&C - \frac{1}{2}[(1+0.8^2)y_t^2 - 2*0.8(y_{t+1}+y_{t-1})y_t]\\
=&C - \frac{1}{2}1.64[y_t - \tfrac{0.8(y_{t+1}+y_{t-1})}{1.64}]^2
\end{aligned}$$

  showing that

$$p(y_t|y_{t-1}, y_{t+1}) = N(\tfrac{1}{2.05}(y_{t+1}+y_{t-1}), 1/1.64).$$

  The reason why we do not get a mean that is exactly the average of y_{t+1} and y_{t-1} is due to that we have a prior on y_t that is centered at zero.

Exercise 4.1

- We have

$$\begin{aligned}\log p(\theta|y) =& C + \sum_{i=1}^{5} \log[1 + (y_i - \theta)^2] \\ \frac{\partial}{\partial\theta} \log p(\theta|y) =& -2 \sum_{i=1}^{5} \frac{(y_i - \theta)}{1 + (y_i - \theta)^2} \\ \frac{\partial^2}{\partial\theta^2} \log p(\theta|y) =& -2 \sum_{i=1}^{5} \frac{-[1 + (y_i - \theta)^2] + 2(y_i - \theta)^2}{[1 + (y_i - \theta)^2]^2} \\ =& 2 \sum_{i=1}^{5} \frac{1 - (y_i - \theta)^2}{[1 + (y_i - \theta)^2]^2}\end{aligned}$$

Exercise 4.15

(a) Define the 50% interval to be $[\hat{\theta}^l, \hat{\theta}^u]$. We have

$$\Pr(\hat{\theta}^l < \theta < \hat{\theta}^u|y) = 0.5$$

Then

$$\begin{aligned}\Pr(\hat{\theta}^l < \theta < \hat{\theta}^u) &= \int_y \Pr(\hat{\theta}^l < \theta < \hat{\theta}^u|y)p(y)dy \\ &= \int_y 0.5p(y)dy = 0.5\end{aligned}$$

(b) For the model, we have, using equations (2.9)-(2.10) that $p(\theta|y) = N(\frac{4}{5}y, \sqrt{\frac{4}{5}}^2)$ which gives a 50% credibility interval of $[\frac{4}{5}y - z_{0.75}\sqrt{\frac{4}{5}}, \frac{4}{5}y + z_{0.75}\sqrt{\frac{4}{5}}]$.

If $\theta = \theta_0$, we have

$$\begin{aligned}&\Pr(\tfrac{4}{5}y - z_{0.75}\sqrt{\tfrac{4}{5}} < \theta_0 < \tfrac{4}{5}y + z_{0.75}\sqrt{\tfrac{4}{5}}|\theta = \theta_0) \\ &\quad= \Pr(y - z_{0.75}\sqrt{\tfrac{5}{4}} < \tfrac{5}{4}\theta_0 < y + z_{0.75}\sqrt{\tfrac{5}{4}}|\theta = \theta_0) \\ &\quad= \Pr(y - \theta_0 - z_{0.75}\sqrt{\tfrac{5}{4}} < \tfrac{1}{4}\theta_0 < y - \theta_0 + z_{0.75}\sqrt{\tfrac{5}{4}}|\theta = \theta_0) \\ &\quad= \Pr(\tfrac{1}{4}\theta_0 - z_{0.75}\sqrt{\tfrac{5}{4}} < y - \theta_0 < \tfrac{1}{4}\theta_0 + z_{0.75}\sqrt{\tfrac{5}{4}}|\theta = \theta_0) \\ &\quad= \Phi(\frac{1}{4}\theta_0 + z_{0.75}\sqrt{\tfrac{5}{4}}) - \Phi(\frac{1}{4}\theta_0 - z_{0.75}\sqrt{\tfrac{5}{4}})\end{aligned}$$

For $\theta_0 = 1$ we get coverage 0.535

(c) See `Exer4_15.R`

Execise 7.3

(a) Under H_0, all observations are independent with marginal distributions $y_j \sim N(0, A^2 + \sigma_j^2)$ giving

$$p(\boldsymbol{y}|H_1) = \frac{1}{(2\pi)^{J/2} \prod_{j=1}^{J}(A^2 + \sigma_j^2)} \exp\left(-\frac{1}{2} \sum_{j=1}^{J} \frac{1}{A^2 + \sigma_j^2} y_j^2\right)$$

Under H_2 we have

$$\begin{aligned}
p(\boldsymbol{y}|H_2) &= \int_\theta p(\boldsymbol{y}|\theta)p(\theta|H_2)d\theta \\
&= \int_\theta p(\theta|H_2)\prod_{j=1}^{J} p(y_j|\theta)d\theta \\
&= \int_\theta \frac{1}{(2\pi)^{(J+1)/2}A\prod_{j=1}^{J}\sigma_j}\exp\left(-\frac{1}{2}[A^{-2}\theta^2+\sum_{j=1}^{J}\sigma_j^{-2}(y_j-\theta)^2]\right)d\theta \\
&= \int_\theta \frac{1}{(2\pi)^{(J+1)/2}A\prod_{j=1}^{J}\sigma_j}\exp\left(-\frac{1}{2}[(A^{-2}+\sum_{j=1}^{J}\sigma_j^{-2})\theta^2-2\sum_{j=1}^{J}\sigma_j^{-2}y_j+\sum_{j=1}^{J}\sigma_j^{-2}y_j^2]\right)d\theta \\
&= \frac{(A^{-2}+\sum_{j=1}^{J}\sigma_j^{-2})^{-1/2}}{(2\pi)^{J/2}A\prod_{j=1}^{J}\sigma_j}\exp\left(-\frac{1}{2}[\sum_{j=1}^{J}\sigma_j^{-2}y_j^2-\left(\frac{\sum_{j=1}^{J}\sigma_j^{-2}y_j}{(A^{-2}+\sum_{j=1}^{J}\sigma_j^{-2})}\right)^2]\right)\times \\
&\quad \int_\theta \frac{1}{(2\pi)^{1/2}(A^{-2}+\sum_{j=1}^{J}\sigma_j^{-2})^{-1/2}}\exp\left(-\frac{(A^{-2}+\sum_{j=1}^{J}\sigma_j^{-2})}{2}[\theta-\frac{\sum_{j=1}^{J}\sigma_j^{-2}y_j}{(A^{-2}+\sum_{j=1}^{J}\sigma_j^{-2})}]^2\right)d\theta \\
&= \frac{1}{(2\pi)^{J/2}(1+A^2\sum_{j=1}^{J}\sigma_j^{-2})^{1/2}\prod_{j=1}^{J}\sigma_j}\exp\left(-\frac{1}{2}[\sum_{j=1}^{J}\sigma_j^{-2}y_j^2-\frac{(A^2\sum_{j=1}^{J}\sigma_j^{-2}y_j)^2}{(1+A^2\sum_{j=1}^{J}\sigma_j^{-2})}]\right)
\end{aligned}$$

Exercise 7.4

$$\begin{aligned}
H_1 : p(\boldsymbol{y}|\boldsymbol{\theta}) &= \prod_{j=1} = N(y_j|\theta_j,\sigma_j^2), & p(\theta_j) &\overset{ind}{=} N(0,A^2) \\
H_2 : p(\boldsymbol{y}|\theta) &= \prod_{j=1} = N(y_j|\theta,\sigma_j^2), & p(\theta) &= N(0,A^2)
\end{aligned}$$

(a) Consider first model H_1. Then marginally $y_j \sim N(0,\sigma_j^2+A^2)$ and all y_j's are independent so that the marginal density becomes

$$\begin{aligned}
p(\boldsymbol{y}|A) &= \prod_{j=1}^{J}\frac{1}{\sqrt{2\pi}\sqrt{\sigma_j^2+A^2}}\exp(-\tfrac{1}{2(\sigma_j^2+A^2)}y_j^2) \\
&= \frac{1}{(2\pi)^{J/2}}\left[\prod_{j=1}^{J}\frac{1}{\sqrt{\sigma_j^2+A^2}}\right]\exp(-0.5\sum_{j=1}^{J}\frac{y_j^2}{\sigma_j^2+A^2})
\end{aligned}$$

Consider then model H_2. In that case

$$
\begin{aligned}
p(\boldsymbol{y}|A) &= \int_\theta \frac{1}{\sqrt{2\pi}A}\exp(-\tfrac{1}{2A^2}\theta^2)\left[\prod_{j=1}^{J}\tfrac{1}{\sqrt{2\pi}\sigma_j}\right]\exp(-\tfrac{1}{2\sigma_j^2}(y_j-\theta)^2)d\theta \\
&= \left[\prod_{j=1}^{J}\frac{1}{\sqrt{2\pi}\sigma_j}\right]\exp(-\tfrac{1}{2}\sum_{j=1}^{J}\frac{y_j^2}{\sigma_j^2})\times \\
&\quad \int_\theta \tfrac{1}{\sqrt{2\pi}A}\exp(-\tfrac{1}{2}[(\tfrac{1}{A^2}+\sum_{j=1}^{J}\tfrac{1}{\sigma_j^2})\theta^2 - 2\sum_{j=1}^{J}\tfrac{y_j}{\sigma_j^2}\theta)])d\theta \\
&= \left[\prod_{j=1}^{J}\frac{1}{\sqrt{2\pi}\sigma_j}\right]\exp(-\tfrac{1}{2}\sum_{j=1}^{J}\frac{y_j^2}{\sigma_j^2})\times \\
&\quad \int_\theta \tfrac{1}{\sqrt{2\pi}A}\exp(-\tfrac{1}{2}(\tfrac{1}{A^2}+\sum_{j=1}^{J}\tfrac{1}{\sigma_j^2})[\theta^2 - 2\frac{\sum_{j=1}^{J}\frac{y_j}{\sigma_j^2}}{(\frac{1}{A^2}+\sum_{j=1}^{J}\frac{1}{\sigma_j^2})}\theta])d\theta \\
&= \left[\prod_{j=1}^{J}\frac{1}{\sqrt{2\pi}\sigma_j}\right]\exp(-\tfrac{1}{2}\sum_{j=1}^{J}\frac{y_j^2}{\sigma_j^2})\frac{1}{A\sqrt{\frac{1}{A^2}+\sum_{j=1}^{J}\frac{1}{\sigma_j^2}}}\exp\left(\frac{1}{2}\left(\frac{(\sum_{j=1}^{J}\frac{y_j}{\sigma_j^2})^2}{(\frac{1}{A^2}+\sum_{j=1}^{J}\frac{1}{\sigma_j^2})}\right)\right)\times \\
&\quad \int_\theta \frac{\sqrt{\frac{1}{A^2}+\sum_{j=1}^{J}\frac{1}{\sigma_j^2}}}{\sqrt{2\pi}}\exp(-\tfrac{1}{2}(\tfrac{1}{A^2}+\sum_{j=1}^{J}\tfrac{1}{\sigma_j^2})[\theta - \frac{\sum_{j=1}^{J}\frac{y_j}{\sigma_j^2}}{(\frac{1}{A^2}+\sum_{j=1}^{J}\frac{1}{\sigma_j^2})}]^2)d\theta \\
&= \left[\prod_{j=1}^{J}\frac{1}{\sqrt{2\pi}\sigma_j}\right]\exp(-\tfrac{1}{2}\sum_{j=1}^{J}\frac{y_j^2}{\sigma_j^2})\frac{1}{A\sqrt{\frac{1}{A^2}+\sum_{j=1}^{J}\frac{1}{\sigma_j^2}}}\exp\left(\frac{1}{2}\left(\frac{(\sum_{j=1}^{J}\frac{y_j}{\sigma_j^2})^2}{(\frac{1}{A^2}+\sum_{j=1}^{J}\frac{1}{\sigma_j^2})}\right)\right)
\end{aligned}
$$

This gives the Bayes factor

$$
\begin{aligned}
BF(H_2;H_1) =& \frac{p(\boldsymbol{y}|H_2}{p(\boldsymbol{y}|H_1)} \\
=& \frac{\prod_{j=1}^{J}\frac{1}{\sqrt{2\pi}\sigma_j}\exp(-\frac{1}{2}\sum_{j=1}^{J}\frac{y_j^2}{\sigma_j^2})\frac{1}{\sqrt{1+\sum_{j=1}^{J}\frac{A^2}{\sigma_j^2}}}\exp\left(\frac{(\sum_{j=1}^{J}\frac{y_j}{\sigma_j^2})^2}{2(\frac{1}{A^2}+\sum_{j=1}^{J}\frac{1}{\sigma_j^2})}\right)}{\frac{1}{(2\pi)^{J/2}}\prod_{j=1}^{J}\frac{1}{\sqrt{\sigma_j^2+A^2}}\exp(-0.5\sum_{j=1}^{J}\frac{y_j^2}{\sigma_j^2+A^2})} \\
=& \frac{\prod_{j=1}^{J}\sqrt{\sigma_j^2+A^2}}{\sqrt{1+\sum_{j=1}^{J}\frac{A^2}{\sigma_j^2}}}\times \\
& \frac{\prod_{j=1}^{J}\frac{1}{\sigma_j}\exp(-\frac{1}{2}\sum_{j=1}^{J}\frac{y_j^2}{\sigma_j^2})\exp\left(\frac{(\sum_{j=1}^{J}\frac{y_j}{\sigma_j^2})^2}{2(\frac{1}{A^2}+\sum_{j=1}^{J}\frac{1}{\sigma_j^2})}\right)}{\prod_{j=1}^{J}\exp(-0.5\sum_{j=1}^{J}\frac{y_j^2}{\sigma_j^2+A^2})} \\
\overset{A \text{ large}}{\approx}& A^{J-1}\frac{1}{\sum_{j=1}^{J}\frac{1}{\sigma_j^2}}\left[\prod_{j=1}^{J}\frac{1}{\sigma_j}\right]\exp(-\frac{1}{2}\sum_{j=1}^{J}\frac{y_j^2}{\sigma_j^2})\exp\left(\frac{(\sum_{j=1}^{J}\frac{y_j}{\sigma_j^2})^2}{2(\sum_{j=1}^{J}\frac{1}{\sigma_j^2})}\right)
\end{aligned}
$$

Exercise 7.5

Model

$$y^{(\phi)} \sim N(\mu, \sigma^2)$$
$$y^{(\phi)} = \frac{y^\phi - 1}{\phi}$$

with prior $p(\mu, \sigma, \phi) = p(\phi)\sigma^{-1}$

(a) We have that (assuming $\phi > 0$)

$$\frac{\partial}{\partial y_i} y_i^{(\phi)} = y_i^{\phi-1}$$
$$p(y_i|\mu, \sigma, \phi) = \frac{1}{\sqrt{2\pi}\sigma} \exp(-\frac{1}{2\sigma^2}(\phi^{-1}(y_i^\phi - 1) - \mu)^2) y_i^{\phi-1}$$

Further,

$$\prod_{i=1}^n p(y_i|\mu, \sigma, \phi) \propto \sigma^{-(n-1)} \exp(-\frac{n}{2\sigma^2} S_2^{(\phi)}) \prod_{i=1}^n y_i^{\phi-1} \times \frac{1}{\sqrt{2\pi}\sigma/n} \exp(-\frac{1}{2\sigma^2/n}(\mu - \bar{y}^{(\phi)})^2)$$

where $\bar{y}^{(\phi)} = n^{-1} \sum_{i=1}^n y_i^{(\phi)}$ and $S_2^{(\phi)} = n^{-1} \sum_{i=1}^n (y_i^{(\phi)} - \bar{y}^{(\phi)})^2$. This gives (using $p(\mu, \sigma^2, \phi) = p(\phi)/\sigma$)

$$p(\mu, \sigma^2, \phi|\boldsymbol{y}) \propto p(\phi)\sigma^{-n} \exp(-\frac{n}{2\sigma^2} S_2^{(\phi)}) \prod_{i=1}^n y_i^{\phi-1} \times \frac{1}{\sqrt{2\pi}\sigma/n} \exp(-\frac{1}{2\sigma^2/n}(\mu - \bar{y}^{(\phi)})^2),$$
$$p(\sigma^2, \phi|\boldsymbol{y}) \propto p(\phi)\sigma^{-n} \exp(-\frac{n}{2\sigma^2} S_2^{(\phi)}) \prod_{i=1}^n y_i^{\phi-1}),$$
$$p(\phi|\boldsymbol{y}) \propto p(\phi) \frac{1}{[S_2^{(\phi)}]^{(n-1)/2}} \prod_{i=1}^n y_i^{\phi-1}$$

Now if $\tilde{y}_i = k y_i$ we obtain

$$\tilde{S}_2^{(\phi)} = k^\phi S_2^{(\phi)}$$
$$\prod_{i=1}^n \tilde{y}_i^{\phi-1} = k^{\phi-1} \prod_{i=1}^n y_i$$
$$p(\phi|\tilde{\boldsymbol{y}}) \propto p(\phi) \frac{1}{[S_2^{(\phi)}]^{(n-1)/2}} \prod_{i=1}^n y_i^{\phi-1} \times \frac{k^{\phi-1}}{k^{\phi(n-1)/2}}$$

giving a very different posterior.

(b) We have in this case

$$p(\phi|\boldsymbol{y}) \propto \frac{1}{[S_2^{(\phi)}]^{(n-3)/2}} \prod y_i^{(\phi-1)(1-1/n)}$$

Further, for $\tilde{y}_i = k y_i$ we obtain

$$\dot{z}^{1-\phi} = \dot{y}^{1-\phi} k^{1-\phi}$$

which will eliminate the $k^{\phi-1}$ factor from earlier (but what about the denominator?).

Exercise 10.2

(a) Define $\mu = E(\theta|y)$. Then

$$\begin{aligned}\hat{\mu}_{MC} =& \frac{1}{S}\sum_{s=1}^{S}\theta^s \\ \text{sd}[\hat{\mu}_{MC}] =& \frac{4}{\sqrt{S}}\end{aligned}$$

For $S = 100$, we get $\text{sd}[\hat{\mu}_{MC}] = 0.4$.

(b) We want $\frac{4}{\sqrt{S}} = 0.1$, giving $S = 40^2 = 1600$.

Exercise 11.5

(a) Since all quantities involved are differences, we can without loss of generality assume $E[\psi_{ij}] = 0$. Let $\sigma^2 = \text{var}[\psi_{ij}]$ and $\tau^2 = \text{Var}[\bar{\psi}_{\cdot j}]$. We then have

$$\begin{aligned}E[W] =& E[s_j^2] = \frac{1}{n-1}E[\sum_{i=1}^{n}\psi_{ij}^2 - n\bar{\psi}_{\cdot j}] \\ =& \frac{1}{n-1}[n\sigma^2 - n\tau^2]\end{aligned}$$

Further, we directly obtain $E[B] = n\tau$ giving

$$E[\widehat{\text{var}}^+[\psi|y]] = \frac{n-1}{n}\frac{1}{n-1}[n\sigma^2 - n\tau^2] + \tau^2 = \sigma^2$$

which leads to the result.

(b) We need that τ^2 is finite, that is, since

$$\tau = n^{-1}\sigma^2 + n^{-1}\sigma^2\sum_{h=1}^{\infty}\rho(h)$$

we need $\sum_{h=1}^{\infty}\rho(h)$ to be finite.

Execise 11.6

(a) We have, since we are running m independent chains that

$$mn\text{var}(\psi_{\cdot\cdot}) = n\text{var}(\psi_{\cdot j}) \tag{1}$$

Further,

$$\text{var}(\psi_{\cdot j}) = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{i'=1}^{n}\text{cov}(\psi_{ij}, \psi_{i'j})$$

For i, i' large we have $\text{cov}(\psi_{ij}, \psi_{i'j}) \approx \sigma^2\rho(|i - i'|)$ giving (assuming n is large enough to ignore the larger errors for the first elements)

$$\begin{aligned}\text{var}(\psi_{\cdot j}) \approx& \frac{1}{n^2}\sum_{i=1}^{n}\sum_{i'=1}^{n}\sigma^2\rho(|i - i'|) = \frac{1}{n^2}\sigma^2[1 + 2\sum_{t=1}^{n}(n-t)\rho(t)] \\ \approx& \frac{1}{n^2}\sigma^2[1 + 2\sum_{t=1}^{n}n\rho(t)] = \frac{1}{n}\sigma^2[1 + \sum_{t=0}^{\infty}\rho(t)]\end{aligned}$$