Proposed solution to the exam in STK4060 & STK9060
Spring 2006

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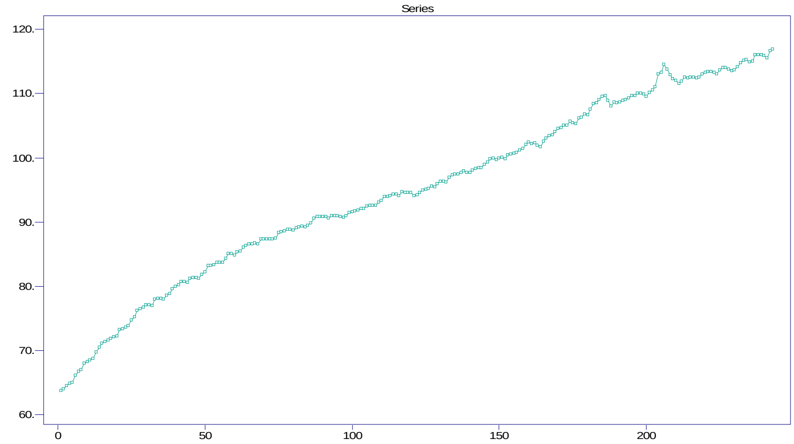
NOTE: Several of the questions in the test have no unique answer; there will always be a subjective element, in particular in selecting the "best" model. Other alternatives than the ones I present here may be seen as equally good, if the argumentation is solid.

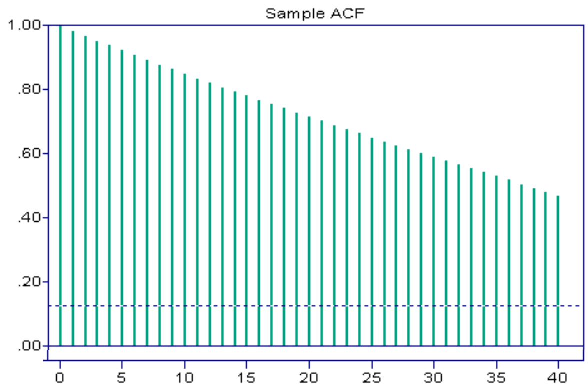
Ex. 1

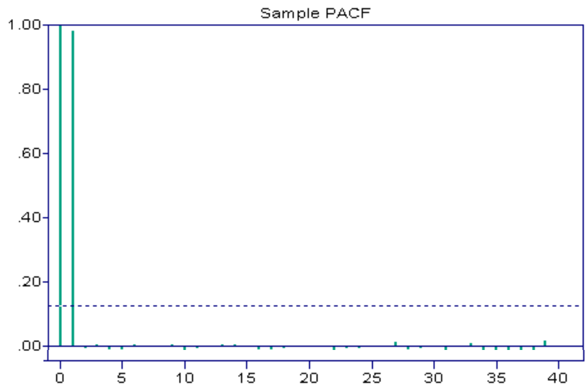
Data: KPI from SSB from Jan. 1986 to March 2006, 243 monthly observations. The data are shown below. A clear, nearly linear trend is the main feature of the data. There seems to be some peculiar “humps” around observation no. 200.



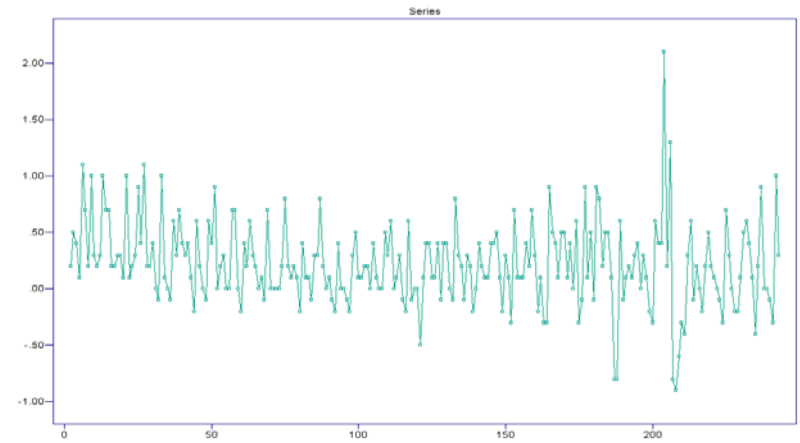
a)

This clear trend, combined with the very persistent ACF shown below shows a clear need for differentiation.



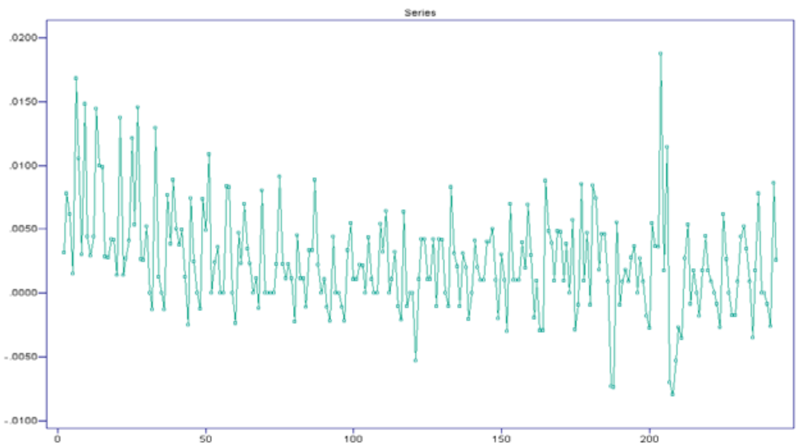


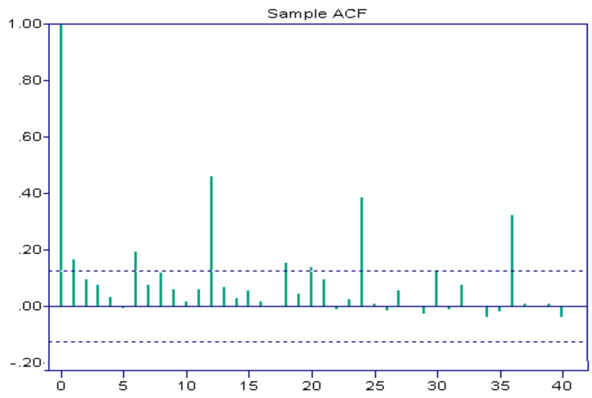
The series after a lag 1 differentiation is shown to the right.

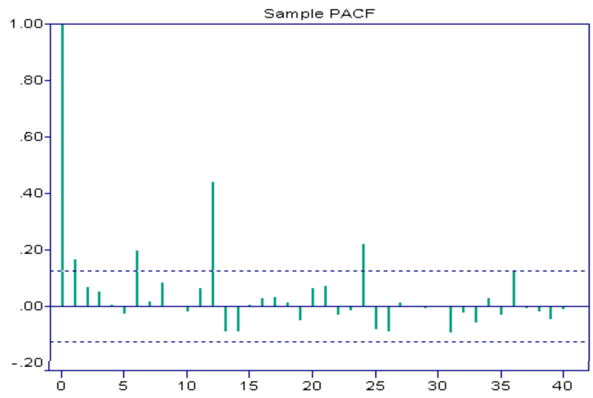


The most distinct feature is the rather erratic behaviour between observations 203 and 209, i.e. November 2002 to July 2003, and a slightly larger variability towards the end of the series. Thus vaguely suggest a log-transformation of the data, although the need for a transformation is by no means obvious. A log-transformation implies that the KPI varies on a relative rather than an absolute scale, which is consistent with economic theory.

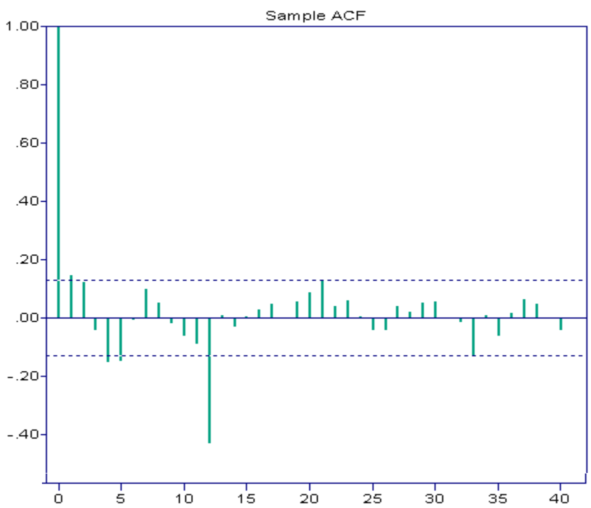
The 1. difference of the log-transformed data is shown to the right, and show the same features.

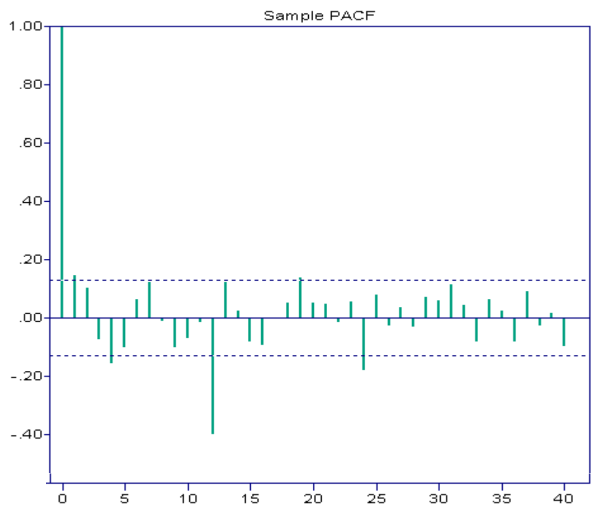






The highly significant, slowly decaying peaks in the ACF at lags 12, 24 and 36 stand out in the figure above, showing a need for differentiation at lag 12 as well. Below are shown the ACF and PACF after differentiation both at lag 1 and lag 12.





There are a number of borderline significant values in the ACF and PACF, but the negative peak at lag 12 stands out. Ignoring all lags but 12, 24 and 36, the patterns in the ACF and PACF are consistent with a SARIMA$(0,1,0)\text{x}(0,1,1)_{12}$ model with seasonal MA-parameter around –0.5. Thus, the first model proposal is this SARIMA$(0,1,0)\text{x}(0,1,1)_{12}$ model. Preliminary estimation, using the innovations method in ITSM, gives $\Theta_{12} = -0.53$, and the results from ML estimation are shown to the right:

The Θ_{12} value of –0.77 is significant. A plot of the residuals (not shown) shows some rather extreme values in early 2003, as expected from the visual inspection of the series. The most extreme is almost 6

```
ARMA Model:
X(t) = Z(t) + .0000 Z(t-1) + .0000 Z(t-2) + .0000 Z(t-3)
     + .0000 Z(t-4) + .0000 Z(t-5) + .0000 Z(t-6) + .0000 Z(t-7)
     + .0000 Z(t-8) + .0000 Z(t-9) + .0000 Z(t-10) + .0000 Z(t-11)
     - .7651 Z(t-12)

WN Variance = .000010

MA Coefficients
        .000000        .000000        .000000        .000000
        .000000        .000000        .000000        .000000
        .000000        .000000        .000000       -.765115


Standard Error of MA Coefficients
        .000000        .000000        .000000        .000000
        .000000        .000000        .000000        .000000
        .000000        .000000        .000000        .152819


(Residual SS)/N = .00000991829

AICC = -.196483E+04
BIC  = -.199131E+04

-2Log(Likelihood) = -.196888E+04
```

standard deviations, and way outside the natural variation. This value alone can easily result in some “spurious” residual autocorrelations.

The residual ACF has two (barely) significant values at lags 1 and 2, a cluster of significant or near significant vales around lag 20, as well as some other scattered near significant values. The Ljung-Box test for randomness rejects the white noise hypothesis at level 0.1%.

Refining the model to a SARIMA(0,1,2)x$(0,1,1)_{12}$ model, where the non-seasonal MA-parameters are expected to be fairly small, gives the model to the right. Note that the non-seasonal parameters are significant, and that the estimates for the seasonal parameter has a much smaller standard deviation compared to the previous model. The AICC criterion has improved as well, while the BIC has become slightly worse.

```
ARMA Model:
X(t) = Z(t) + .1822 Z(t-1) + .1844 Z(t-2) + .0000 Z(t-3)
     + .0000 Z(t-4) + .0000 Z(t-5) + .0000 Z(t-6) + .0000 Z(t-7)
     + .0000 Z(t-8) + .0000 Z(t-9) + .0000 Z(t-10) + .0000 Z(t-11)
     - .7324 Z(t-12) - .1334 Z(t-13) - .1351 Z(t-14)

WN Variance = .000010

MA Coefficients
       .182188       .184448       .000000       .000000
       .000000       .000000       .000000       .000000
       .000000       .000000       .000000      -.732352
      -.133426      -.135081

Standard Error of MA Coefficients
       .064414       .067693       .000000       .000000
       .000000       .000000       .000000       .000000
       .000000       .000000       .000000       .090898
       .000000       .000000

(Residual SS)/N = .00000971260

AICC = -.197756E+04
BIC  = -.198759E+04

-2Log(Likelihood) = -.198574E+04
```

For this model the residual ACF and PACF show no clear patterns. There are a few scattered near-significant values, but the Ljung-Box statistic is not significant even at the 45% level. The only statistic to shed some doubt on the white noise hypothesis is the rank test, which rejects the hypothesis at the 1% level.

Conclusion: The SARIMA(0,1,2)x$(0,1,1)_{12}$ with the parameters from above is an acceptable model for the KPI.

b)

Forecasts for the period Jan. 2005-March 2006 based on data throughout Dec. 2004:

Using ITSM with the above model gives the forecasts in the table to the right:

		95% pred. bounds		
	Prediction	Lower	Upper	Actual
Jan. 2005	113.7	113.0	114.5	113.6
Feb. 2005	114.1	113.0	115.2	113.7
March 2005	114.2	112.7	115.6	114.2
April 2005	114.0	112.3	115.8	114.8
May 2005	113.9	111.9	115.9	115.2
June 2005	113.8	111.6	116.1	115.3
July 2005	113.5	111.1	115.9	114.9
Aug. 2005	113.2	110.6	115.8	115.1
Sep. 2005	113.7	111.0	116.5	116.0
Oct. 2005	113.8	110.9	116.8	116.0
Nov. 2005	113.9	110.8	117.0	116.0
Dec. 2005	114.1	110.9	117.4	115.9
Jan. 2006	114.1	110.7	117.6	115.6
Feb. 2006	114.5	110.9	118.2	116.6
March 2006	114.5	110.7	118.5	116.9

After the first three months, the predictions fall consistently below the actual values. This is probably caused by the fall in the KPI in the first half of 2002. This has resulted in a flattening of the local trend, and the model has not yet “recovered” and picked up the increase.

The above forecasting and comparison is done “within sample”. This means that the data that we forecast have also been used to identify the model and estimate its parameters. Thus we will expect a better fit to the actual observations than we would get from an "out of sample" prediction.

c)

The KPI values are given with only one decimal. Thus a published value of X will actually be X+ε, where ε is the rounding error, uniformly distributed over [-0.05,0.05>. Thus a lower limit for the 1-step prediction error st.dev. (or any prediction error st.dev. for that matter) is the st.dev. of ε. The st.dev. of a uniformly distributed variable is given by its range/√12. In this case the range is 0.1, and the st.dev. is $0.1/\sqrt{12} \approx 0.029$.

d)

Annual inflation (in %) defined as $Y_t = (X_t - X_{t-12})/X_{t-12} \cdot 100\%$, where X_t is the KPI value for month t and Y_t is the % change over the last 12 months.

We have from the exercise text that $Y_t \approx (1-B^{12})\ln(X_t) \cdot 100\%$, and assume the approximation to be exact in the following.

In part a) we found that $(1-B)(1-B^{12})\ln(X_t)=(1+\theta_1 B+\theta_2 B^2)(1+\Theta_{12}B^{12})Z_t$. where Z_t is white noise with variance σ^2, with the parameters given in part a). But since $(1-B^{12})\ln(X_t) = Y_t/100$, we have $(1-B)\,Y_t=(1+\theta_1 B+\theta_2 B^2)(1+\Theta_{12}B^{12})U_t$, where now U_t is white noise with variance $100^2\sigma^2$. Thus, Y_t follows a SARIMA$(0,1,2)\text{x}(0,0,1)_{12}$ model with parameters as found in part a).

Using ITSM with this model, we find the following predictions, based on data up to March 2006. From the predictions we can find the standard error, e.g. as (Upper – Prediction)/1.96. The probability that the actual value will exceed 2.5% is then 1-Φ((2.5-Prediction)/St.err.), where Φ is the standard cumulative normal distribution.

		95% pred. bounds			
	Pred.	Lower	Upper	St. err.	Pr(>2.5%)
Dec. 2006	1.96	-0.51	4.42	1.26	33.3 %
Dec. 2007	1.80	-1.18	4.78	1.52	32.2 %

Note that in the analysis above we have assumed the mean of the differentiated series to be 0. Subtracting the small and insignificant, but negative mean will introduce a deterministic negative trend, which will have a significant impact on the long-term forecasts for the inflation rate.

Ex. 2

Industrial mixing process, ref. details in the exercise text

Notation:

I_t = Total actual (correct) input weight to batch no. t (unobserved)
μ_{I} = Required total input weight according to the recipe, so that $\mathrm{E}I_t = \mu_{\mathrm{I}}$ (known)
Z_{It} = Total input weight error, so that $Z_{It} = I_t - \mu_{\mathrm{I}}$ (unobserved)
σ_I = Standard deviation for Z_{It}. (Known, calculated as the square root of the sum of the variances for the individual weighing processes)

R_t = Residue in the blender after discharging batch no. t (unobserved)
R_t is assumed to follow a stationary AR(1) process: $(R_t - \mu_R) = \varphi(R_{t-1} - \mu_R) + Z_{Rt}$,
where Z_{Rt} WN$(0, \sigma_{\mathrm{R}}^2)$ and the mean μ_R is known.

O_t = Actual output weight from batch no. t (unobserved)

Y_t = Measured output weight from batch no. t (Observed). The weighting equipment is unbiased, so that $\mathrm{E}(Y_t | O_t) = O_t$
Z_{Yt} = Measurement error on output, i.e. $Z_{Yt} = Y_t - O_t$ (unobserved)
σ_Y = Standard deviation for Z_{Yt} : $\sigma_Y = \mathrm{Stdv}(Y_t | O_t)$. (known)

The dynamics of this industrial process can then be written:

$$
\begin{aligned}
I_{t+1} &= \mu_I + Z_{I\,t+1} \\
O_{t+1} &= I_{t+1} + R_t - R_{t+1} \\
R_{t+1} - \mu_R &= \varphi(R_t - \mu_R) + Z_{R\,t+1} \\
Y_{t+1} &= O_{t+1} + Z_{Y\,t+1}
\end{aligned}
\tag{1}
$$

This industrial process can be formulated as a *state-space* model

$$
\begin{aligned}
\mathbf{X_{t+1}} &= \mathbf{F}\mathbf{X_t} + \mathbf{V_{t+1}} \\
Y_t &= \mathbf{G}\mathbf{X_t} + W_t
\end{aligned}
\tag{2}
$$

with state vector $\mathbf{X_t}=(1, I_t, O_t, R_t)'$.

NB: Unfortunately there is typing error in the sign of the R_{t+1} term in the second line of eq. (1) in the exercise text; it should be -, not + ! Also, in the second line of eq. (2) the subscript of $\mathbf{X_t}$ was erroneously given as t+1 in the exercise text.

a)
Repeated use of eq. (1) gives:

$$X_{t+1}=\begin{bmatrix}1\\I_{t+1}\\O_{t+1}\\R_{t+1}\end{bmatrix}=\begin{bmatrix}1\\ \mu_I+Z_{It+1}\\ \mu_I-(1-\varphi)\mu_R+(1-\varphi)R_t+Z_{It+1}-Z_{Rt+1}\\ (1-\varphi)\mu_R+\varphi R_t+Z_{Rt+1}\end{bmatrix}=\underbrace{\begin{bmatrix}1&0&0&0\\ \mu_I&0&0&0\\ \mu_I-(1-\varphi)\mu_R&0&0&1-\varphi\\ (1-\varphi)\mu_R&0&0&\varphi\end{bmatrix}}_{\mathbf{F}}\underbrace{\begin{bmatrix}1\\I_t\\O_t\\R_t\end{bmatrix}}_{\mathbf{X_t}}+\underbrace{\begin{bmatrix}0\\Z_{It+1}\\Z_{It+1}-Z_{Rt+1}\\Z_{Rt+1}\end{bmatrix}}_{\mathbf{V_{t+1}}}$$

$$=\mathbf{FX_t}+\mathbf{V_{t+1}}$$

$$Y_t=O_t+Z_{Yt}=\underbrace{(0,0,0,1)}_{\mathbf{G}}\underbrace{\begin{bmatrix}1\\I_t\\O_t\\R_t\end{bmatrix}}_{\mathbf{X_t}}+\underbrace{Z_{Yt}}_{W_t}=\mathbf{GX_t}+W_t$$

Thus we have the matrices **F** and **G**, and the vectors **X**, **V**, *Y* and *W* as given in the text.

The form and shape of the matrices and vectors in (2), with **F** and **G** independent of time, are more than sufficient for a state-space representation. In addition, (i): $\{\mathbf{V_t}\}$ must be white noise, (ii): $\{W_t\}$ must be white noise, and (iii): $\{\mathbf{V_t}\}$ and $\{W_t\}$ independent of each other.
It is reasonable to assume that the measurement errors, both on the input and on the output, are independent over time. Then $\{Z_{It}\}$ and $\{Z_{Yt}\}$ will be white noise sequences. Furthermore, since $\{Z_{Rt}\}$ is the noise in an AR(1) model, it will be white by definition.
Now, since $\mathbf{V_t}$ only contain elements that are linear combinations of Z_I and Z_R from time *t* only, and $\{Z_{It}\}$and $\{Z_{Rt}\}$are white noise, (i) is fulfilled. Similarly, $\{W_t\}\equiv\{Z_{Yt}\}$, which is white noise by definition, showing (ii). And assuming the measurement error in the output to be independent of those in the input as well as the residue noise we obtain (iii).
Finally, we must assume that the initial state $\mathbf{X_1}$ is independent of the noise processes $\{\mathbf{V_t}\}$ and $\{W_t\}$.

b)
The state-space representation (2) is stable if all the eigenvalues are strictly within the unit circle. Straightforward calculations give that $|\mathbf{F}-\lambda\mathbf{I}| = \lambda^2(1-\lambda)(\varphi-\lambda)$, so the eigenvalues are 0,1 and φ. Since one of the eigenvalues always = 1, the representation is never stable, according to the strict definition. However, the eigenvalue = 1 is related to the constant 1 in the state vector. As long as $|\varphi|<1$, it can be shown that

$$\lim_{n\to\infty}\mathbf{F}^n=\begin{bmatrix}1&0&0&0\\ \mu_I&0&0&0\\ \mu_I&0&0&0\\ \mu_R&0&0&0\end{bmatrix}$$

, so that the only information from the history that persists is the parameter values associated with the first, constant element (=1) in the state vector. If we define *stable* to mean that the impact of previous observation vanishes with time, the state space representation is stable if $|\varphi|<1$.

c)

The φ parameter controls the degree of persistence in the residue process.

φ near +1 implies a strong persistence, so that high, or low, amounts of residue will tend to cluster in time. This could be the case if the amount of residue is to some extent controlled by temperature, humidity or other persistent processes, or the technical condition of the blender. For φ =1, R_t will be a simple random walk.

φ near –1 implies that a high amount of residue in one batch will tend to be followed by a low amount in the next batch, and vice versa. This could be the case if the efficiency of the emptying process depends on the volume in the batch, so that the process becomes more efficient (giving less residue) if the amount in the blender is large, and similarly the other way around. If so, the hypothesis of independence between the input process and the residue process becomes somewhat dubious, though.

For φ =-1, R_t will be an oscillating random walk around the mean level μ_R.

φ near 0 means that the residue process is near white noise, where the residue after each batch is nearly uncorrelated with previous and later values.

For φ = 0, R_t-μ_R is white noise.

d)

ARMA process for $\{O_t\}$.

From (1) we have:

$$O_t = I_t - (1-B)R_t = \mu_I + Z_{It} - (1-B)(\frac{Z_{Rt}}{1-\varphi B} + \mu_R) \qquad (3)$$
$$(1-\varphi B)(O_t - \mu_I) = Z_{It} - \varphi Z_{It-1} - Z_{Rt} + Z_{Rt-1}$$

The left hand side shows an AR(1) structure. The right hand side is 1-correlated, and thus equivalent to an MA(1)-structure. Thus we have that the $\{O_t\}$ process is an ARMA(1,1):

$$(1-\varphi B)(O_t - \mu_I) = (1+\theta B)Z_t. \qquad (4)$$

The AR parameter φ is the same as in the residue process. θ and the white noise variance σ^2 must be found by equating $\gamma(0)$ and $\gamma(1)$ for the right hand sides of the two representations.

From (3) we find:	while (4) gives:
$\gamma(0) = (1+\varphi^2)\sigma_I^2 + 2\sigma_R^2$	$\gamma(0) = (1+\theta^2)\sigma^2$
$\gamma(1) = -\varphi\sigma_I^2 - \sigma_R^2$	$\gamma(1) = \theta\sigma^2$

θ and σ^2 can then be found by solving the equation:

$\dfrac{\theta}{1+\theta^2} = \dfrac{-\varphi\sigma_I^2 - \sigma_R^2}{(1+\varphi^2)\sigma_I^2 + 2\sigma_R^2}$ for θ, using the invertible root, and then finding $\sigma^2 = -(\varphi\sigma_I^2 + \sigma_R^2)/\theta$

As $\varphi \to 1$, the equation for θ will approach $\dfrac{\theta}{1+\theta^2} = -\tfrac{1}{2}$, giving $\theta = -1$. Then the AR and MA terms both become (1-B). They can then be cancelled out, and $\{O_t - \mu_I\}$ becomes white noise with variance $\sigma_I^2 + \sigma_R^2$.

As $\varphi \to 0$, the equation for θ will approach $\dfrac{-\theta}{1+\theta^2} = \dfrac{1}{\tau+2}$, where $\tau = \dfrac{\sigma_R^2}{\sigma_I^2}$, giving $\theta = -\tfrac{1}{2}(\tau + 2 - \sqrt{\tau^2 + 4\tau})$ and $\sigma^2 = -\sigma_R^2/\theta$. The negative sign for the square root must be chosen so that $|\theta| < 1$, i.e.in the invertible region. When τ is small, so that the residue variance is much smaller than the input measurement variance, $\theta \to -1$ and the process approaches a non-invertible MA(1) with variance σ_R^2. When τ is large, so that the residue variance is much larger than the input measurement variance, $\theta \to 0$ and the process approaches white noise with variance σ_I^2.

e)
ARMA prosess for $\{Y_t\}$.

We have from (1) that $Y_t = O_t + Z_{Yt}$. Using the results from d), we have that:

$$Y_t - \mu_I = O_t - \mu_I + Z_{Yt} = \frac{1+\theta B}{1-\varphi B} Z_t + Z_{Yt} \quad \Rightarrow$$
$$(1-\varphi B)(Y_t - \mu_I) = Z_t + \theta Z_{t-1} + Z_{Yt} - \varphi Z_{Yt-1} \tag{5}$$

The parameter θ and the white noise series Z_t are as found in d). Again the right hand side is 1-correlated, and we have an ARMA(1,1) model for $\{Y_t\}$:

$(1-\varphi B)(Y_t - \mu_I) = (1+\omega B)U_t$, where U_t is W.N. with variance ν^2 (6)

We use the same technique as in d).

From (5) we find:

$$\gamma(0) = (1+\theta^2)\sigma^2 + (1+\varphi^2)\sigma_Y^2$$
$$\gamma(1) = \theta\sigma^2 - \varphi\sigma_Y^2 \tag{7}$$

while (6) gives:

$$\gamma(0) = (1+\omega^2)\nu^2$$
$$\gamma(1) = \omega\nu^2 \tag{8}$$

ω and ν^2 can then, as in d), be found by solving the equation:

$\dfrac{\omega}{1+\omega^2} = \dfrac{\theta\sigma^2 - \varphi\sigma_Y^2}{(1+\theta^2)\sigma^2 + (1+\varphi^2)\sigma_Y^2}$ for ω, using the invertible root, and then finding $\nu^2 = (\theta\sigma^2 - \varphi\sigma_Y^2)/\omega$, provided $\omega \neq 0$. If $\omega = 0$ we obtain directly $\nu^2 = \gamma(0) = (1+\theta^2)\sigma^2 + (1+\varphi^2)\sigma_Y^2$

Alternatively, we have from (3) and (1) that

$$(1-\varphi B)(O_t - \mu_I) = Z_{It} - \varphi Z_{It-1} - Z_{Rt} + Z_{Rt-1} \Rightarrow$$
$$(1-\varphi B)(Y_t - Z_{Yt} - \mu_I) = Z_{It} - \varphi Z_{It-1} - Z_{Rt} + Z_{Rt-1} \Rightarrow$$
$$(1-\varphi B)(Y_t - \mu_I) = Z_{It} - \varphi Z_{It-1} - Z_{Rt} + Z_{Rt-1} + Z_{Yt} - \varphi Z_{Yt-1}$$

Thus, as an alternative to (7), we have:

$$\gamma(0) = (1+\varphi^2)\sigma_I^2 + 2\sigma_R^2 + (1+\varphi^2)\sigma_Y^2$$
$$\gamma(1) = -\varphi\sigma_I^2 - \sigma_R^2 - \varphi\sigma_Y^2 \tag{9}$$

ω and ν^2 can then, as before, be found by solving the equation:

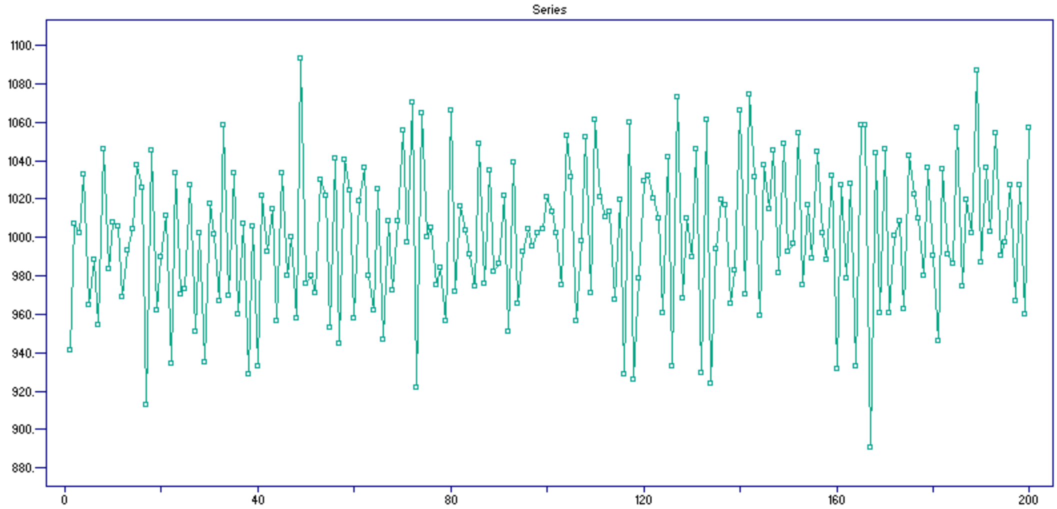
$\dfrac{\omega}{1+\omega^2} = \dfrac{-\varphi\sigma_I^2 - \sigma_R^2 - \varphi\sigma_Y^2}{(1+\varphi^{2)}\sigma_I^2 + 2\sigma_R^2 + (1+\varphi^2)\sigma_Y^2}$ for ω, using the invertible root,

and then finding $\nu^2 = (-\varphi\sigma_I^2 - \sigma_R^2 - \varphi\sigma_Y^2)/\omega$, provided $\omega \neq 0$.

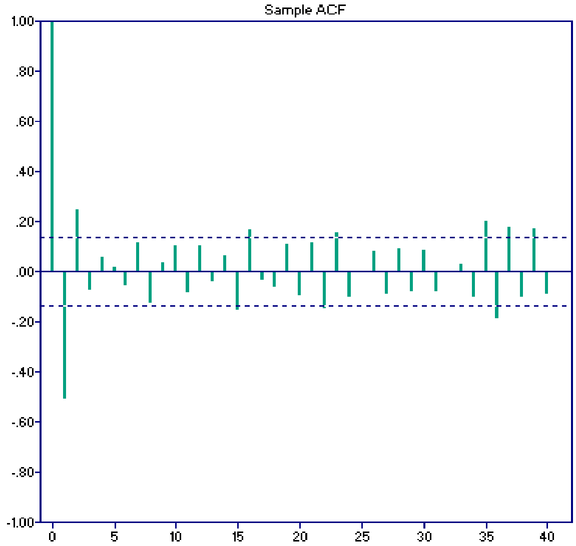
If $\omega = 0$ we obtain directly $\nu^2 = \gamma(0) = (1+\varphi^2)\sigma_I^2 + 2\sigma_R^2 + (1+\varphi^2)\sigma_Y^2$ (10)

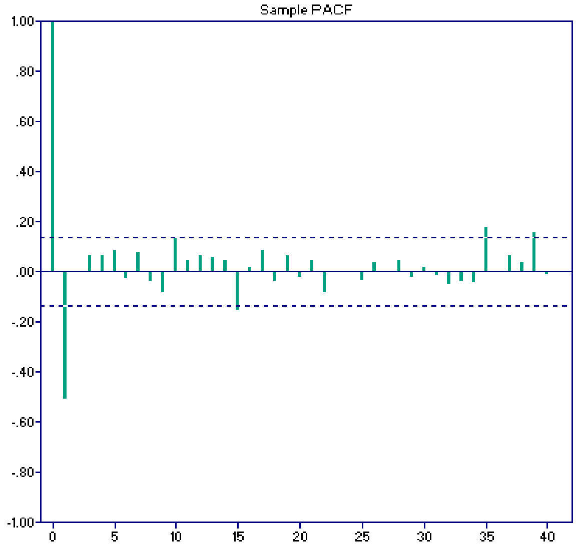
f)

200 observations of Y_t.



The plot shows a rather erratic behavior around a fairly stable mean value at appr. 1000.





The ACF and PACF are consistent with an AR(1) model with parameter ca. –0.5. There is no sign of any need for any MA parameter. Estimation of an AR(1) model gives the model to the right:

```
ARMA Model:
X(t) = - .5142 X(t-1)
     + Z(t)

WN Variance = .109650E+04

AR Coefficients
     -.514167

Standard Error of AR Coefficients
       .061228

(Residual SS)/N = .109650E+04

AICC = .197192E+04
BIC  = .197281E+04
FPE  = .110752E+04

-2Log(Likelihood) = .196786E+04
```

An analysis of the residuals shows no sign of model inadequacy, and there does not seem to be any room for further model improvement.

Enforcing an MA(1) term gives, as expected, θ very close to 0, and far from significant. Thus we continue with the simple AR(1) model, which is consistent with the results from e) with

φ=-0.51, ω=0 and ν^2=1097.

Assuming $\omega=0$, we get from (10):

$$\nu^2 = (1+\varphi^2)\sigma_I^2 + 2\sigma_R^2 + (1+\varphi^2)\sigma_Y^2 \Rightarrow \sigma_R^2 = \tfrac{1}{2}(\nu^2 - (1+\varphi^2)(\sigma_I^2 + \sigma_Y^2))$$

With φ = -0.51, $\nu^2 = 1097$ from the model estimation, and with $\sigma_I^2 = \sigma_Y^2 = 400$ from the text, we obtain $\sigma_R^2 = 42.5 \Rightarrow \sigma_R = 6.5$, as requested.

However, inserting these values into (9) we find $\gamma(1) = -\varphi\sigma_I^2 - \sigma_R^2 - \varphi\sigma_Y^2 = 365.5$, which is far from consistent with the lack of an MA term in the identified model for the data. This may indicate that there is a printing error in the exercise text, so that $\sigma_I^2 \neq 400$ or $\sigma_Y^2 \neq 400$. Again, using (9) and requiring $\gamma(1)$=0, we obtain:

$$\begin{array}{l} 1097 = (1+(-.51)^2)\sigma_I^2 + 2\sigma_R^2 + (1+(-.51)^2)\sigma_Y^2 \\ 0 = .51\sigma_I^2 - \sigma_R^2 - .51\sigma_Y^2 \end{array} \Rightarrow \begin{array}{l} \sigma_I^2 + \sigma_Y^2 = 481 \\ \sigma_R^2 = 245 \end{array}.$$

Thus the correct solution, consistent with the data, is likely to be σ_R=15.7 and σ_I=20 / σ_Y=9 or σ_I=9 / σ_Y=20.