SOLUTION TO EXERCISE 4.4

$N_i(t)$ has intensity process

$\lambda_i(t) = Y_i(t)\{\beta_0(t) + \beta_1(t)x_i\}$, where $Y_i(t)$ is an at risk indicator and $x_i = 0,1$ is a group indicator. The number of risk in group 0 (i.e. $x_i = 0$) is

$Y^{(0)}(t) = \sum_{i=1}^{n}(1-x_i)Y_i(t)$ and the number of risk in group 1 (i.e. $x_i = 1$) is

$Y^{(1)}(t) = \sum_{i=1}^{n} x_i Y_i(t)$. Thus $Y_\cdot(t) = Y^{(0)}(t) + Y^{(1)}(t)$

$= \sum_{i=1}^{n} Y_i(t)$.

a) We have that

$$X(t) = \begin{pmatrix} Y_1(t) & Y_1(t)x_1 \\ Y_2(t) & Y_2(t)x_2 \\ \vdots & \\ Y_n(t) & Y_n(t)x_n \end{pmatrix}$$

By a direct computation we get

$$X(t)^T X(t) = \begin{pmatrix} \sum_{i=1}^{n} Y_i(t)^2 & \sum_{i=1}^{n} Y_i(t)^2 x_i \\ \sum_{i=1}^{n} Y_i(t)^2 x_i & \sum_{i=1}^{n} Y_i(t)^2 x_i^2 \end{pmatrix}$$

$$= \begin{pmatrix} Y_\bullet(t) & Y^{(1)}(t) \\ Y^{(1)}(t) & Y^{(1)}(t) \end{pmatrix}$$

where the last equality follows since $Y_i(t)^2 = Y_i(t)$ and $x_i^2 = x_i$.

b) We have that

$$\text{Det}\left(X(t)^T X(t)\right) = Y_\bullet(t)\, Y^{(1)}(t) - Y^{(1)}(t)^2$$

$$= Y^{(1)}(t)\left(Y_\bullet(t) - Y^{(1)}(t)\right)$$

$$= Y^{(0)}(t)\, Y^{(1)}(t)$$

It follows that (when $Y^{(0)}(t) > 0$ and $Y^{(1)}(t) > 0$):

$$\left(X(t)^T X(t)\right)^{-1} = \frac{1}{\operatorname{Det}\left(X(t)^T X(t)\right)} \begin{pmatrix} Y^{(1)}(t) & -Y^{(1)}(t) \\ -Y^{(1)}(t) & Y_\cdot(t) \end{pmatrix}$$

$$= \frac{1}{Y^{(0)}(t)Y^{(1)}(t)} \begin{pmatrix} Y^{(1)}(t) & -Y^{(1)}(t) \\ -Y^{(1)}(t) & Y^{(0)}(t)+Y^{(1)}(t) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{Y^{(0)}(t)} & -\frac{1}{Y^{(0)}(t)} \\ -\frac{1}{Y^{(0)}(t)} & \frac{1}{Y^{(0)}(t)} + \frac{1}{Y^{(1)}(t)} \end{pmatrix}$$

c) The estimator $\hat{B}(t) = (\hat{B}_0(t), \hat{B}_1(t))^T$ is given by [cf. (4.58) & (4.59) in ABG]:

$$\hat{B}(t) = \int_0^t J(u)\left(X(u)^T X(u)\right)^{-1} X(u)^T \, dN(u)$$

where $N(t) = (N_1(t), \ldots, N_n(t))^T$.

Now we have that

$$(X(t)^T X(t))^{-1} X(t)^T$$

$$= \begin{pmatrix} \frac{Y_1(t)}{Y^{(0)}(t)} - \frac{Y_1(t)x_1}{Y^{(0)}(t)} & \cdots\cdots & \frac{Y_n(t)}{Y^{(0)}(t)} - \frac{Y_n(t)x_n}{Y^{(0)}(t)} \\ \frac{Y_1(t)x_1}{Y^{(0)}(t)} + \frac{Y_1(t)x_1}{Y^{(1)}(t)} - \frac{Y_1(t)}{Y^{(0)}(t)} & \cdots\cdots & \frac{Y_n(t)x_n}{Y^{(0)}(t)} + \frac{Y_n(t)x_n}{Y^{(1)}(t)} - \frac{Y_n(t)}{Y^{(0)}(t)} \end{pmatrix}$$

Hence we get

$$\hat{B}(t) = \int_0^t J(u) \begin{pmatrix} \sum_{i=1}^n \frac{(1-x_i)Y_i(u)}{Y^{(0)}(u)} dN_i(u) \\ \sum_{i=1}^n \frac{x_i Y_i(u)}{Y^{(1)}(u)} dN_i(u) - \sum_{i=1}^n \frac{(1-x_i)Y_i(u)}{Y^{(0)}(u)} dN_i(u) \end{pmatrix}$$

$$= \int_0^t J(u) \begin{pmatrix} dN^{(0)}(u)/Y^{(0)}(u) \\ dN^{(1)}(u)/Y^{(1)}(u) - dN^{(0)}(u)/Y^{(0)}(u) \end{pmatrix}$$

where $N^{(0)}(t) = \sum_{i=1}^n (1-x_i) N_i(t)$ and $N^{(1)}(t) = \sum_{i=1}^n x_i N_i(t)$

Thus $\hat{B}_0(t) = \int_0^t (J(u)/Y^{(0)}(u)) dN^{(0)}(u)$ is the Nelson-Aalen estimator for group 0 and $\hat{B}_1(t) = \int_0^t (J(u)/Y^{(1)}(u)) dN^{(1)}(u) - \int_0^t (J(u)/Y^{(0)}(u)) dN^{(0)}(u)$ is the difference between the Nelson-Aalen estimators in the two groups.