Exercise 5.1

The counting processes $N_i(t)$; $i = 1, 2, \dots, n$; have intensity processes of the form $\lambda_i(t) = Y_i(t)e^{\beta}$, where the $Y_i(t)$ are at risk indicators. We introduce $R_i(t) = \int_0^t Y_i(u)\,\mathrm{d}u$ and remember that a 'dot' indicates summation over i.

a) The log likelihood is

$$\begin{aligned}
\ell(\beta) &= \sum_{i=1}^{n} \int_0^{\tau} \log \lambda_i(t)\,\mathrm{d}N_i(t) - \int_0^{\tau} \lambda_{\bullet}(t)\,\mathrm{d}t \\
&= \sum_{i=1}^{n} \int_0^{\tau} \log\{Y_i(t)e^{\beta}\}\,\mathrm{d}N_i(t) - \int_0^{\tau} Y_{\bullet}(t)e^{\beta}\,\mathrm{d}t \\
&= \beta \sum_{i=1}^{n} \int_0^{\tau} \mathrm{d}N_i(t) - e^{\beta} \int_0^{\tau} Y_{\bullet}(t)\,\mathrm{d}t \\
&= \beta N_{\bullet}(\tau) - e^{\beta} R_{\bullet}(\tau).
\end{aligned}$$

The score function is

$$U(\beta) = \ell'(\beta) = N_{\bullet}(\tau) - e^{\beta} R_{\bullet}(\tau)$$

The score equation $U(\beta) = 0$ is solved by

$$e^{\widehat{\beta}} = \frac{N_{\bullet}(\tau)}{R_{\bullet}(\tau)}$$

which gives

$$\widehat{\beta} = \log\left(\frac{N_{\bullet}(\tau)}{R_{\bullet}(\tau)}\right) = \log \widehat{\nu}$$

b) The observed information is

$$I(\beta) = -U'(\beta) = e^{\beta} R_{\bullet}(\tau)$$

and hence

$$I(\widehat{\beta}) = e^{\widehat{\beta}} R_{\bullet}(\tau) = \frac{N_{\bullet}(\tau)}{R_{\bullet}(\tau)} R_{\bullet}(\tau) = N_{\bullet}(\tau)$$

It follows that $\widehat{\beta}$ is approximately normally distributed around the true value of β with a variance that may be estimated by $1/I(\widehat{\beta}) = 1/N_{\bullet}(\tau)$.

c) The standard 95% confidence interval for β is

$$\widehat{\beta} \pm \frac{1.96}{\sqrt{N_{\bullet}(\tau)}}$$

By exponentiating both endpoints, we find that

$$\exp\left\{\widehat{\beta} \pm \frac{1.96}{\sqrt{N_{\bullet}(\tau)}}\right\} = \widehat{\nu} \exp\left\{\pm \frac{1.96}{\sqrt{N_{\bullet}(\tau)}}\right\}$$

is an approximate 95% confidence interval for $\nu = e^{\beta}$.

Exercise 5.4

The counting processes $N_i(t)$; $i = 1, 2, \ldots, n$; have intensity processes of the form $\lambda_i(t) = \nu Y_i(t)$, where the $Y_i(t)$ are at risk indicators. We introduce $R_i(t) = \int_0^t Y_i(u)\, \mathrm{d}u$ and remember that a 'dot' indicates summation over i.

a) The log-likelihood is

$$\begin{aligned}
\ell(\nu) &= \sum_{i=1}^{n} \int_0^\tau \log \lambda_i(t)\, \mathrm{d}N_i(t) - \int_0^\tau \lambda_{\bullet}(t)\, \mathrm{d}t \\
&= \sum_{i=1}^{n} \int_0^\tau \log\{\nu Y_i(t)\}\, \mathrm{d}N_i(t) - \int_0^\tau \nu Y_{\bullet}(t)\, \mathrm{d}t \\
&= \log \nu \int_0^\tau \mathrm{d}N_{\bullet}(t) - \nu \int_0^\tau Y_{\bullet}(t)\, \mathrm{d}t \\
&= (\log \nu)\, N_{\bullet}(\tau) - \nu R_{\bullet}(\tau)
\end{aligned}$$

The score function is

$$U(\nu) = \ell'(\nu) = \frac{N_{\bullet}(\tau)}{\nu} - R_{\bullet}(\tau) \tag{E.1}$$

b) When ν is the true value of the parameter, we know that

$$M_i(t) = N_i(t) - \int_0^t \lambda_i(t)\, \mathrm{d}t = N_i(t) - \nu \int_0^t Y_i(t)\, \mathrm{d}t$$

is a mean zero martingale. By aggregating over i we find that

$$M_{\bullet}(t) = N_{\bullet}(t) - \nu R_{\bullet}(t) \tag{E.2}$$

is a mean zero martingale. From (E.1) we then obtain when ν is the true value of the parameter

$$U(\nu) = \frac{M_{\bullet}(\tau) + \nu R_{\bullet}(t)}{\nu} - R_{\bullet}(\tau) = \frac{M_{\bullet}(\tau)}{\nu} \tag{E.3}$$

Now $\mathrm{E}\{M_{\bullet}(\tau)\} = 0$ and [cf. (2.24) and (2.43) in the ABG-book]

$$\mathrm{Var}\{M_{\bullet}(\tau)\} = \mathrm{E}\langle M_{\bullet}\rangle(\tau) = \mathrm{E}\left\{\int_0^\tau \lambda_{\bullet}(t)\, \mathrm{d}t\right\} = \nu \mathrm{E}\left\{\int_0^\tau Y_{\bullet}(t)\, \mathrm{d}t\right\} = \nu \mathrm{E}\{R_{\bullet}(\tau)\}$$

From (E.3) we then get that $\mathrm{E}\{U(\nu)\} = 0$ and

$$\mathrm{Var}\{U(\nu)\} = \frac{\mathrm{Var}\{M_{\bullet}(\tau)\}}{\nu^2} = \frac{\mathrm{E}\{R_{\bullet}(\tau)\}}{\nu}$$

when ν is the true parameter value.

c) By (E.1) the observed information is

$$I(\nu) = -U'(\nu) = \frac{N_{\bullet}(\tau)}{\nu^2} \tag{E.4}$$

Using (E.2) we may write when ν is the true value of the parameter

$$I(\nu) = \frac{M_{\bullet}(\tau) + \nu R_{\bullet}(\tau)}{\nu^2} = \frac{M_{\bullet}(\tau)}{\nu^2} + \frac{R_{\bullet}(\tau)}{\nu}$$

and by taking expectations, we find $\mathrm{E}\{I(\nu)\} = \mathrm{E}\{R_{\bullet}(\tau)\}/\nu$.