

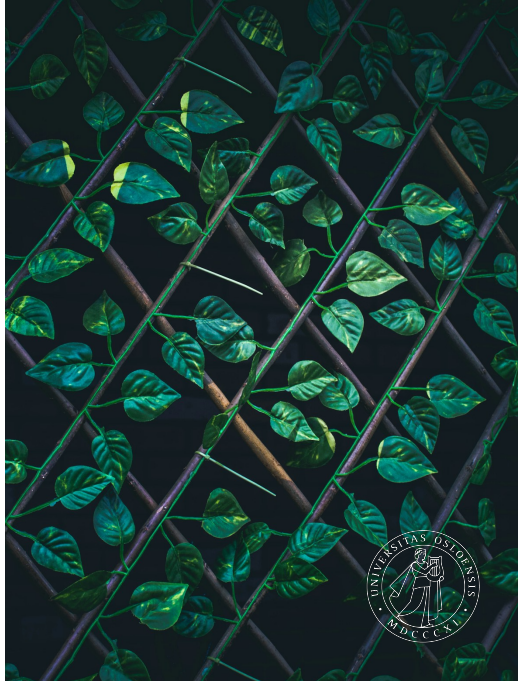
UNIVERSITY OF OSLO

STK4150/9150 - Environmental and
Spatial Statistics

Spatial lattice processes, part II

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Lattice processes

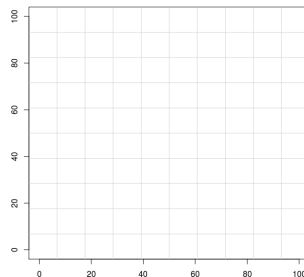
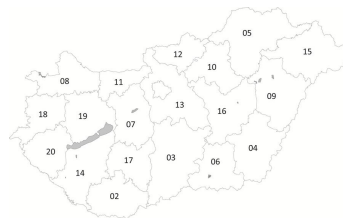
$\{Y(\mathbf{s}) : \mathbf{s} \in D_s\}$, D_s finite (or countable)
and $\mathbf{s}$ (typically) corresponds to an area

Let

$$W_{ij} = \begin{cases} 1 & \text{if } \mathbf{s}_i \text{ and } \mathbf{s}_j \text{ are neighbors} \\ 0 & \text{otherwise} \end{cases}$$

Aim: Construct model based on $\{W_{ij}\}$

We call these *Markov models*



Define $\mathcal{N}(\mathbf{s}_i) = \{j; W_{ij} = 1\}$, the *neighbor set* of site i with $\mathbf{s}_i \in D_S$.

The *conditional autoregressive* (CAR) process is given by

$$\begin{aligned}\mathbb{E}[Y(s_i)|\mathbf{Y}_{-i}] &= \sum_{j \in \mathcal{N}(\mathbf{s}_i)} c_{ij} Y(\mathbf{s}_j) \\ \text{Var}[Y(s_i)|\mathbf{Y}_{-i}] &= \sigma_i^2\end{aligned}$$

If this yields a *legal* simultaneous distribution:

$$\mathbf{Y} \sim \text{Gau}(\mathbf{0}, \mathbf{Q}^{-1})$$

with $Q_{ii} = \sigma_i^{-2}$, $Q_{ij} = -\sigma_i^{-2}c_{ij}$ for $j \neq i$ and $Q_{ij} = Q_{ji}$.

Some basics for defining these models

- $\mathbf{y} \equiv (y(\mathbf{s}_1), \dots, y(\mathbf{s}_n))' \in \mathbb{R}^n$
- $\Omega \equiv \{\mathbf{y} : \mathbb{P}(\mathbf{y}) > 0\}$, $\Omega_i \equiv \{y(\mathbf{s}_i) : \mathbb{P}(y(\mathbf{s}_i)) > 0\}$
- Positivity condition: $\Omega = \Omega_1 \times \dots \times \Omega_n$ (not essential, but simplifies results)
- Besag's Lemma (Brook's Lemma): For $\mathbf{y}, \mathbf{w} \in \Omega$ it holds that

$$\frac{\Pr(\mathbf{y})}{\Pr(\mathbf{w})} = \prod_{i=1}^n \frac{\Pr(y(\mathbf{s}_i) | y(\mathbf{s}_1), \dots, y(\mathbf{s}_{i-1}), w(\mathbf{s}_{i+1}), \dots, w(\mathbf{s}_n))}{\Pr(w(\mathbf{s}_i) | y(\mathbf{s}_1), \dots, y(\mathbf{s}_{i-1}), w(\mathbf{s}_{i+1}), \dots, w(\mathbf{s}_n))}.$$

Gives relationship between conditional and simultaneous distributions
(if simultaneous distribution exist)

- Typically $\mathbf{w}$ fixed (e.g $\mathbf{w} = \mathbf{0}$), gives

$$\Pr(\mathbf{y}) \propto \prod_{i=1}^n \frac{\Pr(y(\mathbf{s}_i) | y(\mathbf{s}_1), \dots, y(\mathbf{s}_{i-1}), w(\mathbf{s}_{i+1}), \dots, w(\mathbf{s}_n))}{\Pr(w(\mathbf{s}_i) | y(\mathbf{s}_1), \dots, y(\mathbf{s}_{i-1}), w(\mathbf{s}_{i+1}), \dots, w(\mathbf{s}_n))}$$

- *Clique*: A set of locations that consists either of a single site or a set of sites that are *all* neighbors to each other

Assume $\mathbb{P}(\mathbf{0}) > 0$. Then the *negpotential function* is defined by $Q(\mathbf{y}) \equiv \log\{\Pr(\mathbf{y})/\Pr(\mathbf{0})\}$.

For example, if $\mathbf{Y} \sim \text{Gau}(\mathbf{0}, \mathbf{Q}^{-1})$, then $Q(\mathbf{y}) = -(1/2)\mathbf{y}^\top \mathbf{Q} \mathbf{y}$ for $\mathbf{y} \in \mathbb{R}^n$.

We also define so-called *G-functions* where it holds that

$$Q(\mathbf{y}) = \sum_i y_i G_i(y_i) + \sum_{ij} y_i y_j G_{ij}(y_i, y_j) + \sum_{ijk} y_i y_j y_k G_{ijk}(y_i, y_j, y_k) \\ + \cdots + y_1 y_2 \cdots y_m G_{1,\dots,m}(y_1, \dots, y_m)$$

Hammersley-Clifford theorem

The negpotantial function Q given by

$$Q(\mathbf{y}) = \sum_i y_i G_i(y_i) + \sum_{ij} y_i y_j G_{ij}(y_i, y_j) + \\ \sum_{ijk} y_i y_j y_k G_{ijk}(y_i, y_j, y_k) + \cdots + y_1 y_2 \cdots y_m G_{1, \dots, m}(y_1, \dots, y_m)$$

satisfies the property: If $\{i_1, i_2, \dots, i_k\}$ is not a clique, then $G_{i_1, i_2, \dots, i_k}(\cdot) \equiv 0$.

Proof of the Hammersley-Clifford theorem

“Not a clique” means there exist two variables that are not neighbors. Assume these are $i_1 = i$ and $i_2 = l$. Define $\mathbf{y}_i = (y_1, \dots, y_{i-1}, 0, y_{i+1}, \dots, y_m)$. Then

$$\frac{\Pr(\mathbf{y})}{\Pr(\mathbf{y}_i)} = \frac{\Pr(y_i | \mathbf{y}_{-i})}{\Pr(0 | \mathbf{y}_{-i})} \quad \text{do not depend on } y_l$$

$$\frac{\Pr(\mathbf{y})}{\Pr(\mathbf{y}_i)} = \exp(Q(\mathbf{y}) - Q(\mathbf{y}_i))$$

It follows that $Q(\mathbf{y}) - Q(\mathbf{y}_i)$ does not depend on y_l .

Proof of the Hammersley-Clifford theorem, part II

$$Q(\mathbf{y}) - Q(\mathbf{y}_i) = y_i G_i(y_i) + \sum_{j \neq i} y_i y_j G_{i,j}(y_i, y_j) + \sum_{j,k \neq i} y_i y_j y_k G_{i,j,k}(y_i, y_j, y_k) + \dots$$

- Setting $y_j = 0$ for $j \neq i, l$ implies $y_i y_l G_{i,l}(y_i, y_l) = 0$
 - For $y_l = 0$ it holds that $y_i y_l G_{i,l}(y_i, y_l) = 0$ for any y_i
 - This implies $y_i y_l G_{i,l}(y_i, y_l) = 0$ for $y_l \neq 0$ and any y_i
 - This again implies $G_{i,l}(y_i, y_l) = 0$
- Similarly: Putting $y_j = 0, j \neq i, k, l$ implies $G_{i,k,l}(y_i, y_k, y_l) = 0$
- etc.
- Implies all G 's containing i, l are zero.

Existence of simultaneous distribution

Besag's lemma and Hammersley-Clifford theorem give results assuming simultaneous distribution exist.

Assume conditional distributions exist and

1. $\Omega = \Omega_1 \times \Omega_2 \times \cdots \times \Omega_m$
2. The G-functions are invariant under permutation
3. $\sum_{\mathbf{t} \in \Omega} \exp\{Q(\mathbf{t})\} < \infty$

Then simultaneous distribution exist.

Condition 1 can be relaxed, see eq (4.116) in the book.

Auto-spatial models

Auto-spatial models: Only second order G -function terms

$$Q(\mathbf{y}) = \sum_i y_i G_i(y_i) + \sum_{ij} y_i y_j G_{ij}(y_i, y_j)$$

This class of models includes

- the CAR model
- latent Gaussian processes
- the auto Poisson model

CAR model with $\boldsymbol{\mu} = \mathbf{0}$

$$\begin{aligned}\log Pr(\mathbf{y}) &= -\frac{m}{2} \log(2\pi) - \frac{1}{2} \log |\boldsymbol{\Sigma}| - \frac{1}{2} \mathbf{y}^T \boldsymbol{\Sigma}^{-1} \mathbf{y} = -\frac{m}{2} \log(2\pi) + \frac{1}{2} \log |\mathbf{Q}| - \frac{1}{2} \mathbf{y}^T \mathbf{Q} \mathbf{y} \\ &= -\frac{m}{2} \log(2\pi) + \frac{1}{2} \log |\mathbf{Q}| - \frac{1}{2} \sum_{ij} y_i Q_{ij} y_j = -\frac{m}{2} \log(2\pi) + \frac{1}{2} \log |\mathbf{Q}| - \frac{1}{2} \sum_i y_i^2 Q_{ii} - \frac{1}{2} \sum_i \sum_{j \in \mathcal{N}_i} y_i Q_{ij} y_j\end{aligned}$$

yields

$$\begin{aligned}\log Pr(\mathbf{0}) &= -\frac{m}{2} \log(2\pi) + \frac{1}{2} \log |\mathbf{Q}| \\ Q(\mathbf{y}) &= -\frac{1}{2} \sum_i y_i^2 Q_{ii} - \frac{1}{2} \sum_i \sum_{j \in \mathcal{N}_i} y_i Q_{ij} y_j \\ G_i(y_i) &= -\frac{1}{2} y_i Q_{ii} \quad G_{ij}(y_i, y_j) = -\frac{1}{2} Q_{ij}, \quad G_{ijk}(y_i, y_j, y_k) = 0\end{aligned}$$

Note: $\{i, j, k\}$ might be a clique but still $G_{ijk}(y_i, y_j, y_k) = 0$.

CAR model with $\mu = 0$

$$Pr(\mathbf{y}) = Pr(\mathbf{0}) e^{\mathbf{Q}(\mathbf{y})}$$

$$\begin{aligned} \mathbf{Q}(\mathbf{y}) &= -\frac{1}{2} \sum_i y_i^2 Q_{ii} - \frac{1}{2} \sum_{i \neq j} y_i Q_{ij} y_j \\ &= -\frac{1}{2} \sum_i y_i^2 Q_{ii} - \frac{1}{2} \sum_i \sum_{j \in \mathcal{N}_i} y_i Q_{ij} y_j \end{aligned}$$

Assume $Q_{ij} = 0$. Then

- $Q_{ji} = 0$
- $i \notin \mathcal{N}_j$ and $j \notin \mathcal{N}_i$
- If $\mathcal{N}_i$ are small for all i , then $\mathbf{Q}$ is *sparse*

Latent Gaussian processes

Assume $\mathbf{y} \sim \text{MVN}(\boldsymbol{\mu}, \mathbf{Q}^{-1})$ ($\mathbf{Q}$ sparse) and $z_i | \mathbf{y} \sim f(z_i | y_i)$ (cond. independent). Then

$$p(\mathbf{y} | \mathbf{z}) \propto p(\mathbf{y}) p(\mathbf{z} | \mathbf{y}) = p(\mathbf{y}) \prod_i p(z_i | y_i)$$

$$\log p(\mathbf{y} | \mathbf{z}) = \text{Const} - \frac{m}{2} \log(2\pi) + \frac{1}{2} \log |\mathbf{Q}| - \frac{1}{2} \sum_{ij} y_i Q_{ij} y_j + \sum_i \log f(z_i | y_i)$$

$$Q(\mathbf{y}) = \log p(\mathbf{y} | \mathbf{z}) - \log p(\mathbf{0} | \mathbf{z}) = -\frac{1}{2} \sum_{ij} y_i Q_{ij} y_j + \sum_i \log f(z_i | y_i) - \sum_i \log f(z_i | 0)$$

$$= -\frac{1}{2} \sum_i y_i^2 Q_{ii} + \sum_i \log f(z_i | y_i) - \sum_i \log f(z_i | 0) - \frac{1}{2} \sum_{ij} y_i Q_{ij} y_j$$

$$G_i(y_i) = -\frac{1}{2} y_i Q_{ii} + \sum_i \log f(z_i | y_i) - \log f(z_i | 0)$$

$$G_{ij}(y_i) = -\frac{1}{2} Q_{ij}$$

Finding G functions and simultaneous distribution

Assume simultaneous distribution exist. Then, for MRFs

$$P(y_i|\mathbf{y}_{-i}) = \frac{P(\mathbf{y})}{P(\mathbf{y}_{-i})} \propto P(\mathbf{y}) \propto \exp(Q(\mathbf{y}))$$

$$\propto \exp\{y_i G_i(y_i) + y_i \sum_j y_j G_{ij}(y_i, y_j) + \dots\}$$

Now assume conditional distributions are specified. Then

- set $y_j = 0, j \neq i$ for finding $G_i(y_i)$
- set $y_k = 0, k \neq i, j$ for finding $G_{ij}(y_i, y_j)$
- etc.

We then get that $P(\mathbf{y}) \propto \exp(Q(\mathbf{y}))$

Auto logistic models with $y_i \in \{0, 1\}$

$$\Pr(y_i | \mathbf{y}_{-i}) = \frac{\exp\{\alpha_i y_i + \sum_j \theta_{ij} y_i y_j\}}{1 + \exp\{\alpha_i y_i + \sum_j \theta_{ij} y_i y_j\}} \propto \exp\left\{\alpha_i y_i + \sum_j \theta_{ij} y_i y_j\right\}$$

$$G_i(y_i) = \alpha_i$$

$$G_{ij}(y_i, y_j) = \theta_{ij}$$

$$Q(\mathbf{y}) = \alpha_i y_i + \sum_{i,j} y_i y_j \theta_{ij}$$

$$P(\mathbf{y}) \propto \exp\left\{\sum_i \alpha_i y_i + \sum_{i,j} y_i y_j \theta_{ij}\right\}$$

On a regular lattice we have the *Ising model*:

$i = (u, v)$, $\mathcal{N}_i = \{(u, v - 1), (u, v + 1), (u - 1, v), (u + 1, v)\}$ and $\theta_{ij} = 0$ for non-neighbors

Auto Poisson models (misprint in eq (4.114) in the book)

With $\theta_{ij} = 0, j \notin \mathcal{N}_i$ we have

$$\begin{aligned} P(y_i | y_{-i}) &= \text{Poisson} \left(\exp \left\{ \alpha_i + \sum_{j=1}^m \theta_{ij} y_j \right\} \right) = \text{Poisson} \left(\exp \left\{ \alpha_i + \sum_{j \in \mathcal{N}_i} \theta_{ij} y_j \right\} \right) \\ &= \exp \left\{ y_i \left(\alpha_i + \sum_{j \in \mathcal{N}_i} \theta_{ij} y_j \right) - \log y_i! - e^{\alpha_i + \sum_{j=1}^m \sum_{j \in \mathcal{N}_i} y_j} \right\} \\ &\propto \exp \left\{ y_i \alpha_i - \log y_i! + y_i \sum_{j \in \mathcal{N}_i} \theta_{ij} y_j \right\} \end{aligned}$$

$$G_i(y_i) = \alpha_i - y_i^{-1} \log y_i!$$

$$G_{ij}(y_i, y_j) = \theta_{ij}$$

$$Q(\mathbf{y}) = \alpha_i y_i - \log y_i + \sum_{i,j} y_i y_j \theta_{ij}$$

Auto models for exponential families

$$f_Y(y|\theta) = \exp\{(y\tilde{\theta} - a(\theta))/\phi + \log c(y; \phi)\}$$

Auto models for these families are (a bit more general in the book)

$$[Y_i|\mathbf{Y}_{-i}] \equiv \exp(A_i(\mathbf{Y}_{-i})(Y_i - B_i) + C_i(Y_i) + D_i(\mathbf{Y}_{-i}))$$

$$\tilde{\theta} = A_i(\mathbf{Y}_{-i}) = \alpha_i + \sum_{j \neq i} \theta_{ij}(Y_j - B_j)$$

$$Q(\mathbf{y}) = \sum_i [\alpha_i(y_i - B_i) + C_i(y_i)] + \sum_{i < j} \theta_{ij}(y_i - B_i)(y_j - B_j)$$

The class includes auto Gaussian (CAR) models, auto gamma models, the Winsorized auto Poisson model, the auto beta model

In summary

- $\mathbf{Y} \sim \text{Gau}(\boldsymbol{\mu}, (\mathbf{I} - \mathbf{C})^{-1}\mathbf{M})$, $\text{Diag}(\mathbf{C}) = \mathbf{0}$, often $\boldsymbol{\mu} = \mathbf{X}^T \boldsymbol{\beta}$
- Building block for more complex models
- Geostatistical model: $\mathbf{Y} \sim \text{Gau}(\boldsymbol{\mu}_Y, \boldsymbol{\Sigma}_Y)$
 $C_Y(\mathbf{s}_j - \mathbf{s}_i) \approx 0 (= 0)$, for $\|\mathbf{s}_j - \mathbf{s}_i\|$ "large" $\Rightarrow \boldsymbol{\Sigma}_Y$ "sparse"
- MRF: $c_{ij} \approx 0 (= 0)$ for $\|\mathbf{s}_j - \mathbf{s}_i\|$ "large" $\Rightarrow \boldsymbol{\Sigma}_Y^{-1}$ "sparse"
- MRF to geostat: $\boldsymbol{\Sigma}_Y = (\mathbf{I} - \mathbf{C})^{-1}\mathbf{M}$
- geostat to MRF: $\mathbf{M} = \text{Diag}(\boldsymbol{\Sigma}_Y^{-1})$, $\mathbf{C} = \mathbf{I} - \mathbf{M}\boldsymbol{\Sigma}_Y^{-1}$
- Differ in "sparsity"
- Differ in how to define dependence, *distance* versus *neighborhood*

We still need to talk about

- requirements for a positive definite and symmetric (legal) covariance/precision matrix
- how to estimate these models

Parameterizations of CAR models with $\Sigma_Y = (\mathbf{I} - \mathbf{C})^{-1} \mathbf{M}$

- $\mathbf{M} = \text{diag}\{\tau_i^2\}$, $\tau_i > 0 \quad \forall i$
- Often
 - $\mathbf{C} = \phi \mathbf{H}$, $\mathbf{H}$ known and/or
 - $\mathbf{M} = \tau^2 \Delta$, Δ known diagonal matrix.
- What requirements are needed on (ϕ, τ) ?

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- What requirements are needed on (ϕ, τ) ?
- Need $\mathbf{M}^{-1} \mathbf{C}$ symmetric:
 - $\mathbf{Q}_Y = \mathbf{M}^{-1} [\mathbf{I} - \mathbf{C}] = \mathbf{M}^{-1} - \mathbf{M}^{-1} \mathbf{C}$ and $\mathbf{M}^{-1}$ is automatically symmetric
 - In general: $\tau_i^{-2} C_{ij} = \tau_j^{-2} C_{ji}$
 - $\mathbf{M} = \tau^2 \Delta$, $\mathbf{C} = \phi \mathbf{H} \Rightarrow \Delta_{ii}^{-1} h_{ij} = \Delta_{jj}^{-1} h_{ji}$
- Also need $\Sigma_Y = (\mathbf{I} - \mathbf{C})^{-1} \mathbf{M}$ positive definite, or (equivalently) $\mathbf{Q} = \mathbf{M}^{-1} (\mathbf{I} - \mathbf{C})$ positive definite

CAR: Requirements for a positive definiteness

- When is $\mathbf{Q} = \mathbf{M}^{-1}(\mathbf{I} - \mathbf{C})$ positive definite?
- Since $\mathbf{M}^{1/2}$ is positive definite ($\tau_i > 0$), this is equivalent to $\mathbf{M}^{-1/2}(\mathbf{I} - \mathbf{C})\mathbf{M}^{1/2} = \mathbf{I} - \mathbf{M}^{1/2}\mathbf{C}\mathbf{M}^{-1/2}$ being positive definite
- Let $\mathbf{A}$ be symmetric ($\mathbf{A} = \mathbf{M}^{1/2}\mathbf{C}\mathbf{M}^{-1/2}$)
 - spectral decomposition: $\mathbf{A} = \mathbf{T}\mathbf{\Lambda}\mathbf{T}^T$ where $\mathbf{T}\mathbf{T}^T = \mathbf{I}$ and $\mathbf{\Lambda}$ contains eigenvalues.
 - $\mathbf{A}$ is positive definite if all eigenvalues are positive.
 - $\mathbf{I} - \mathbf{A} = \mathbf{I} - \mathbf{T}\mathbf{\Lambda}\mathbf{T}^T = \mathbf{T}\mathbf{T}^T - \mathbf{T}\mathbf{\Lambda}\mathbf{T}^T = \mathbf{T}[\mathbf{I} - \mathbf{\Lambda}]\mathbf{T}^T$
 $\Rightarrow \mathbf{I} - \mathbf{\Lambda}$ contain eigenvalues of $\mathbf{I} - \mathbf{A}$.
- If λ eigenvalue of $\mathbf{M}^{-1/2}\mathbf{C}\mathbf{M}^{1/2}$, then $1 - \lambda$ eigenvalue of $\mathbf{I} - \mathbf{M}^{-1/2}\mathbf{C}\mathbf{M}^{1/2}$
- Positive definite if $1 - \lambda > 0$, or $\lambda < 1$.

CAR: Requirements for a positive definiteness (cont)

- Assume $\mathbf{C} = \phi \mathbf{H}$, so that $\mathbf{A} = \phi \mathbf{M}^{1/2} \mathbf{H} \mathbf{M}^{-1/2}$
- $\tilde{\lambda}$ eigenvalue of $\mathbf{M}^{1/2} \mathbf{H} \mathbf{M}^{-1/2}$ gives $\lambda = \phi \tilde{\lambda}$ eigenvalue of $\mathbf{A}$.
- Need $\lambda < 1$ or $\phi \tilde{\lambda} < 1$.
- For $\tilde{\lambda}$ positive, need $\phi < \lambda^{-1}$
- For $\tilde{\lambda}$ negative, need $\phi > \lambda^{-1}$
- Let $\tilde{\lambda}_{(1)} \leq \tilde{\lambda}_{(2)} \leq \dots \leq \tilde{\lambda}_{(n)}$ be ordered eigenvalues
- We have $\tilde{\lambda}_{(1)} < 0, \tilde{\lambda}_{(n)} > 0$ (not obvious!)
- Requirement: $\tilde{\lambda}_{(1)}^{-1} < \phi < \tilde{\lambda}_{(n)}^{-1}$
- If $\mathbf{M} = \tau^2 \Delta$, τ just a scaling factor, same requirements

Inhomogeneous processes

- $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\delta}, \boldsymbol{\delta} \sim \text{Gau}(\mathbf{0}, (\mathbf{I} - \mathbf{C})^{-1}\mathbf{M})$
- For $\mathbf{M} = \tau^2\boldsymbol{\Delta}$, with different diagonal elements, called *inhomogeneous process*.
- Define $\tilde{\mathbf{Y}} = \boldsymbol{\Delta}^{-1/2}\mathbf{Y}, \tilde{\mathbf{X}} = \boldsymbol{\Delta}^{-1/2}\mathbf{X}, \tilde{\boldsymbol{\delta}} = \boldsymbol{\Delta}^{-1/2}\boldsymbol{\delta}$,
- Then $\tilde{\mathbf{Y}} = \tilde{\mathbf{X}}\boldsymbol{\beta} + \tilde{\boldsymbol{\delta}}, \tilde{\boldsymbol{\delta}} \sim \text{Gau}(\mathbf{0}, \tau^2(\mathbf{I} - \tilde{\mathbf{C}})^{-1})$
 $\tilde{\mathbf{C}} = \boldsymbol{\Delta}^{-1/2}\mathbf{C}\boldsymbol{\Delta}^{1/2}$
- Note: $c_{ij} = 0 \Leftrightarrow \tilde{c}_{ij} = 0$
- For $\mathbf{C} = \phi\mathbf{H}$: $\tilde{\mathbf{C}} = \phi\boldsymbol{\Delta}^{-1/2}\mathbf{H}\boldsymbol{\Delta}^{1/2}$

How to model **C** (or **H**)?

- Can show $[\text{cor}(Y_i, Y_j | Y_k, k \neq i, j)]^2 = c_{ij}c_{ji}$
- Need $0 \leq c_{ij}c_{ji} \leq 1$
- In general: c_{ij} can depend on distance between $\mathbf{s}_i$ and $\mathbf{s}_j$.
(distance between center points)
- Regular lattice: $c_{ij} = 0$ for
 - $\|s_i - s_j\| > 1$ (1. order)
 - $\|s_i - s_j\| > \sqrt{2}$ (2. order)
 - $\|s_i - s_j\| > 2$ (2. order)
- Irregular lattice: $c_{ij} = 0$ for
 - i and j do not share border
 - $\|s_i - s_j\| > \text{threshold}$
- Size of c_{ij} : Depending on distance, number of neighbors

Consider now $\mathbf{M} = \tau^2 \Delta$, $\mathbf{C} = \phi \mathbf{H}$. Define $\mathbf{A}$ such that $a_{ij} = I(i \text{ and } j \text{ neighbors})$.

- *Homogeneous CAR* (HCAR)

$$\Delta = \mathbf{I}, \mathbf{H} = \mathbf{A}$$

$$\Rightarrow \text{cor}(Y_i, Y_j | Y_k, k \neq i, j) = \phi$$

Need (at least) $\phi \in (0, 1)$

- *Weighted CAR* (WCAR)

$$\Delta = \text{diag}\{|N(\mathbf{s}_i)|^{-1}\}, h_{ij} = a_{ij}|N(\mathbf{s}_i)|^{-1}$$

$$\Rightarrow \text{cor}(Y_i, Y_j | Y_k, k \neq i, j) = \phi |N(\mathbf{s}_i)|^{-1/2} |N(\mathbf{s}_j)|^{-1/2}$$

Need (at least) $\phi \in (0, \min_{ij} |N(\mathbf{s}_i)|^{-1/2} |N(\mathbf{s}_j)|^{-1/2})$

- *Autocorrelated CAR* (ACAR)

$$\Delta = \text{diag}\{|N(\mathbf{s}_i)|^{-1}\}, h_{ij} = a_{ij}|N(\mathbf{s}_i)|^{-1/2} |N(\mathbf{s}_j)|^{1/2}$$

$$\Rightarrow \text{cor}(Y_i, Y_j | Y_k, k \neq i, j) = \phi$$

Need (at least) $\phi \in (0, 1)$

Spatial lattice processes, part II

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