STK4150/9150 - Environmental and Spatial Statistics

Multivariate extremes

Thordis Thorarinsdottir

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Examples of questions on rare complex events

- How widespread are extreme convective precipitation events that occur in the Oslo region in summer?
- What are the chances of simultaneously experiencing a large flood in a river and one of its side streams or tributaries?
- What are the chances of an extreme event somewhere along the Sørlandet railway line in a given year? (Figures from banenor.no and wikipedia)





The spatial extreme value problem

Assume the phenomena we are interested in can be described as a spatially indexed stochastic process

$$\{Y(s) \, : \, s \in D \subset \mathbb{R}^2\}$$

In order to answer the questions on the previous slide, we then need to characterize the extremal behaviour of $Y(s)$.

Margins and dependence

Think of $Y(s)$ in terms of finite dimensional distributions $\mathbf{Y} = (Y(s_1), \ldots, Y(s_d))$.

Probabilities of extreme multivariate/spatial events are determined by *marginal* and *dependence* features.

Copula construction separates these. Let

$$\mathbb{P}(Y(s_j) \leq y_j) = F_j(y_j), \quad j = 1, \ldots, d$$
$$\mathbb{P}(Y(s_i) \leq y_1, \ldots, Y(s_d) \leq y_d) = F(y_1, \ldots, y_d)$$

Then

$$F(y_1, \ldots, y_d) = C(F_1(y_1), \ldots, F_d(y_d))$$

where $C : [0,1]^d \to [0,1]$ is the *copula*.

The estimation of margins and dependence are both important and often performed separately.

Marginal modelling across space

We are often interested in $Y(s)$ even when not extreme everywhere

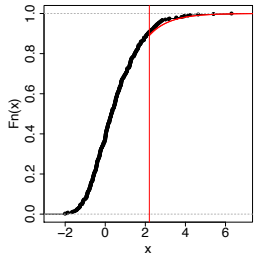
Marginal modelling across space

We are often interested in $Y(s)$ even when not extreme everywhere

- For an arbitrary location s_j and a high threshold u_j, a generalized Pareto distribution above u_j

$$Y(s_j) - u_j | Y(s_j) > u_j \sim \text{GP}(\sigma(s_j), \xi(s_j)).$$

- Extreme value theory tells us nothing about marginal behavior below u_j
- Semi-parametric solution:
  - $\text{GP}(\sigma(s_j), \xi(s_j))$ above u_j
  - empirical distribution function below u_j



Marginal modelling across space, part II

This is a simple easy option

- no need to consider spatial nonstationarity in the margins
- no need for known distributional form in the body

However,

- no borrowing strength or learning important spatial covariates
- no inferring behavior at unobserved locations
- no capturing trends in tail/body/both

Modelling the dependence structure

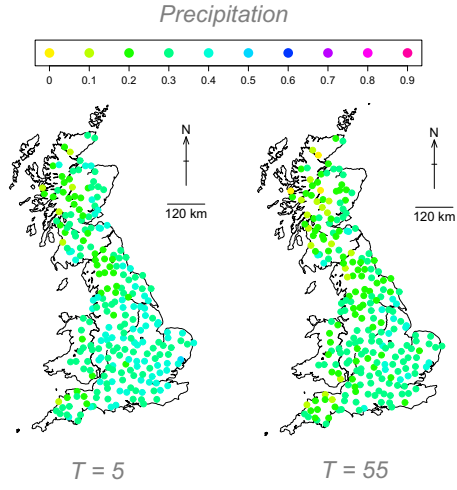
For modelling the dependence structure, we assume common marginal distributions for Y over space.

Then, we need to consider

- how to define spatial extremes
- how to capture extreme dependence at observed levels
- how to extrapolate to higher levels
- how to estimate the models

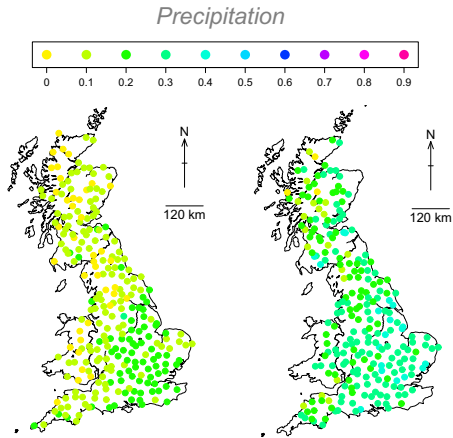
Capturing extreme dependence at observed levels, Ex. 1

Example from Great Britain: Expected proportion of neighboring sites (within a radius of *N* km) that exceed their *p*th quantile (*T* year return level) during an event in which cite *i* exceeds its *p*th quantile (Keef et al., 2009).



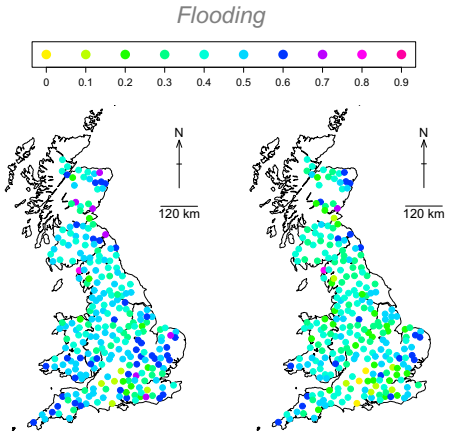
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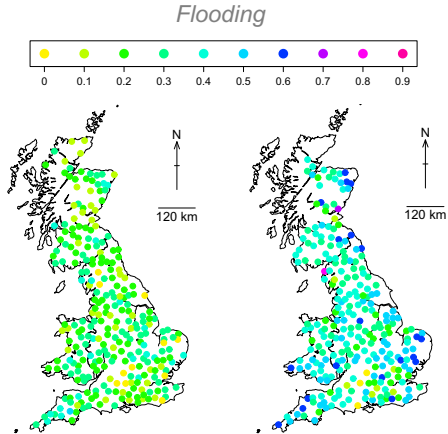
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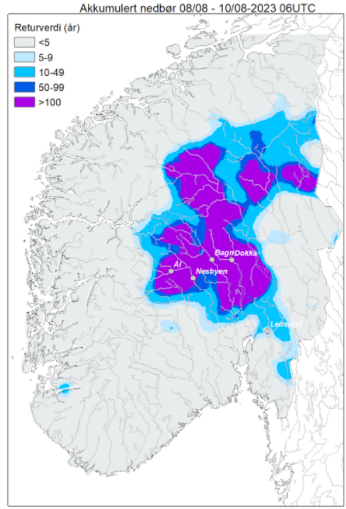
Capturing extreme dependence at observed levels. Ex. 2

Example from Great Britain: Expected proportion of neighboring sites (within a radius of *N* km) that exceed their *p*th quantile (*T* year return level) during an event in which cite *i* exceeds its *p*th quantile (Keef et al., 2009).

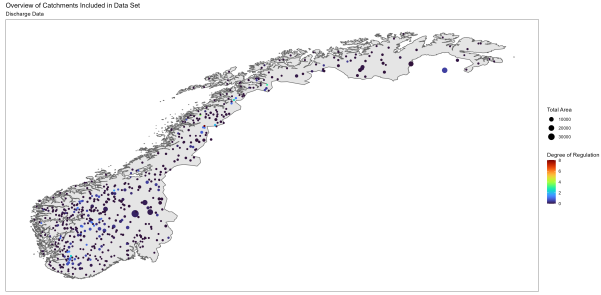


Capturing extreme dependence at observed levels. Ex. 2

What are the chances of simultaneous *large events* at a *certain proportion* of observation stations within a *region*?



Capturing extreme dependence at observed levels. Ex. 2

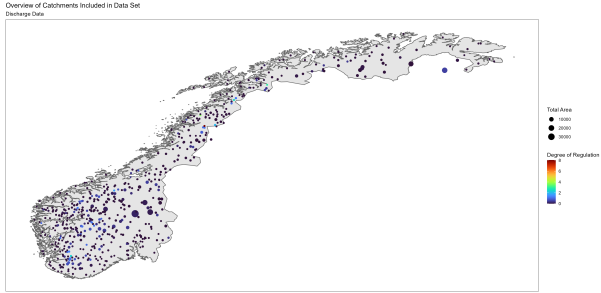


A large event is

a) among the 10% largest events at a location

b) among the 1% largest events at a location

Capturing extreme dependence at observed levels. Ex. 2



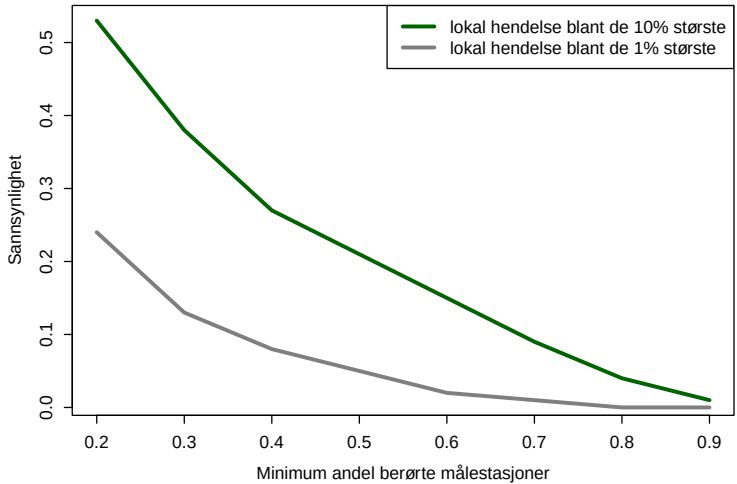
We focus on 38 locations in Østlandet with daily data from 1985-2022.

An event is defined according to two rules:

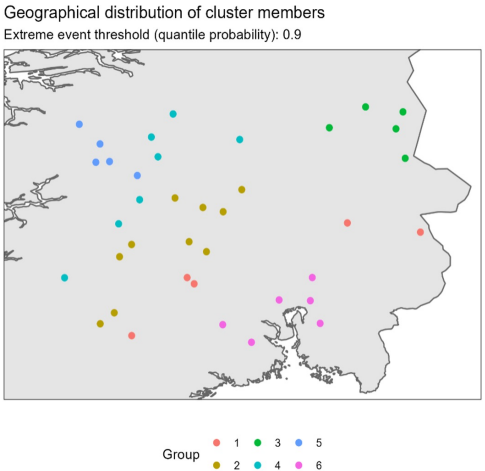
1. We can have only have one event within the area over the course of 7 days
2. An event can be observed up to two days before or after the event date

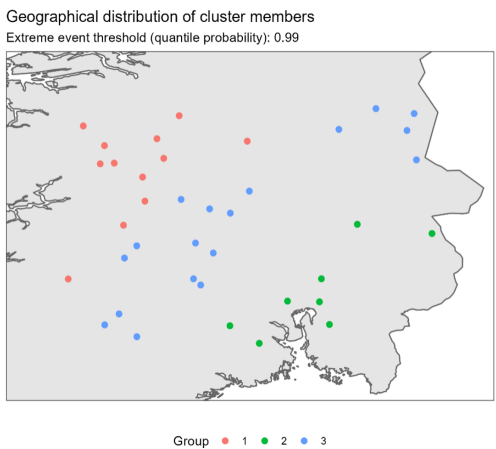
Capturing extreme dependence at observed levels. Ex. 2



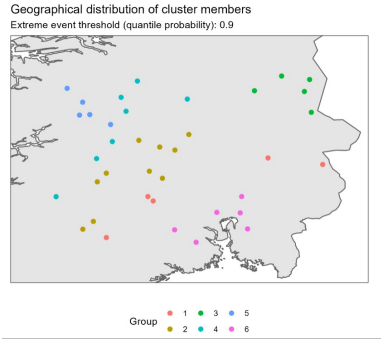


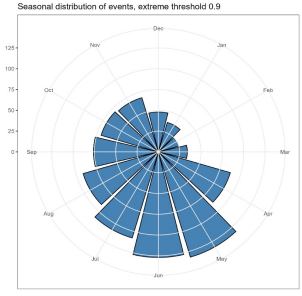
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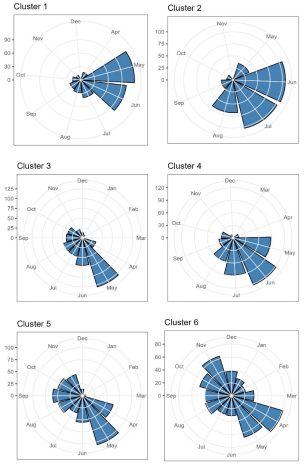




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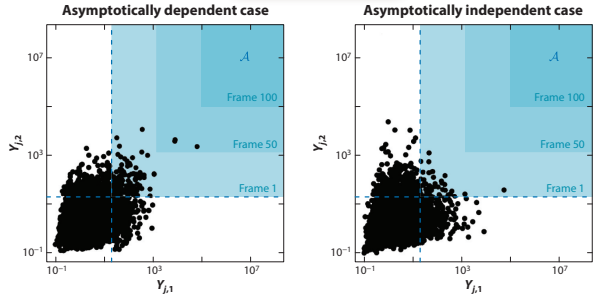


Useful, but not entirely satisfying

While such empirical analyses can be we useful, they are not entirely satisfying

- we can’t extrapolate to unobserved levels
- we can’t extrapolate to unobserved locations within the region
- we don’t have a good description of *why* things are happening the way they are

Asymptotic dependence vs independence

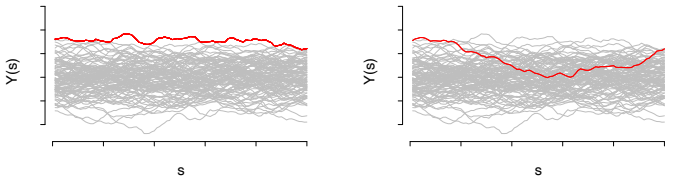


- *Asymptotically independent*: The dependency structure becomes weaker for more extreme values
- *Asymptotically dependent*: The dependency structure stabilizes for more extreme values

Defining spatial extremes

How do we define an extreme value of a spatial process $Y(s)$?

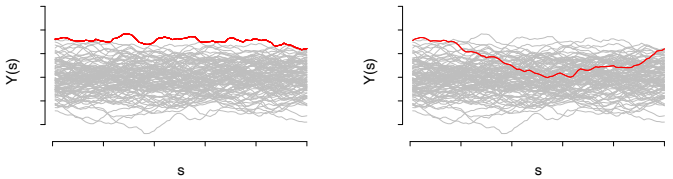
1. Consider *pointwise maxima* over several copies $\max_{1 \leq i \leq n} Y_i(s)$
2. Characterize $\{Y(s) | g(Y(s)) > u\}$ for a *high threshold* u



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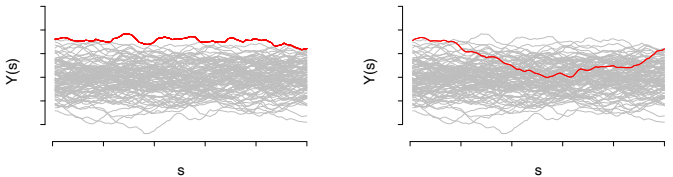


Option 1 leads to *max-stable processes*. However, they possess restrictive dependence structure, inference is hard, and it is not clear if they are relevant for answering our motivational questions.

Defining spatial extremes

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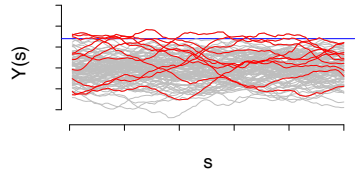
Currently, there is a higher emphasis on modelling *original events*. This is usually more relevant in practice. “Limiting models” have restrictive dependence structures, while “subasymptotic models” are more flexible.

Defining threshold exceedances

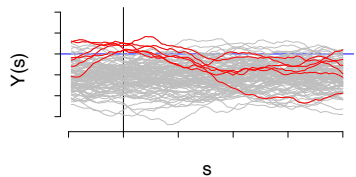
Assume common marginal distributions for Y over space.

Consider $Y(s)|g(Y(s)) > u$ for some function g.

$$g(Y(s)) = \sup_D Y(s) : \ Y(s)|\sup_{s \in D} Y(s) > u$$



$$g(Y(s)) = Y(s_0) : \ Y(s)|Y(s_0) > u$$



Generalized Pareto processes

Pareto processes arise when we consider $Y(s) | \sup_{s \in D} Y(s) > u$.

- Provide a useful definition of a spatial extreme event
- Only suitable for *asymptotic dependent* processes $Y(s)$:

$$\begin{aligned}\chi_q(s_i, s_j) &= \mathbb{P}(Y(s_i) > F_i^{-1}(q) | Y(s_j) > F_j^{-1}(q)) \\ &\to \chi(s_j, s_j) > 0, \quad s_i \neq s_j, \quad q \to 1.\end{aligned}$$

- Consequently, it is assumed that the shape of the event is independent of the level
- However, environmental extremes tend to become *more localized as they become more intense*

Types of spatial extremal dependence

Consider the behavior of

$$\chi_q(s_i, s_j) = \mathbb{P}(Y(s_i) > F_i^{-1}(q) | Y(s_j) > F_j^{-1}(q)), \quad \text{as } q \to 1$$

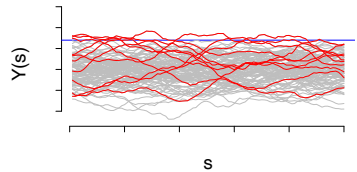
- A general characterization is

$$\chi_q(s_i, s_j) = \chi(s_i, s_j) + K(1-q)^{\gamma}[1 = o(1)], \quad K > 0$$

- This covers all scenarios:
  - $\chi(s_i, s_j) > 0, \gamma \to \infty$: Pareto process
  - $\chi(s_i, s_j) > 0, \gamma \in (0, \infty)$: asymptotic dependence, non-limiting
  - $\chi(s_i, s_j) = 0, \gamma \in (0, \infty)$: asymptotic independence

Subasymptotic tail models

Approximate the "tail" of $Y(s)$ via the tail of another spatial process with flexible extremal dependence properties



One such framework are *random scale models*

$$X(s) = RW(s), \quad R > 0, \text{ independent of } W$$

R controls tails heaviness while W changes extremal dependence properties

- Dependence structure of Y at large values $\sim$ dependence structure of X at large values
- We can also construct models to well represent $\chi_q(s_i, s_j)$

Subasymptotic tail models, part II

Some examples of *random scale models*

$$X(s) = RW(s), \quad R > 0, \text{ independent of } W$$

are

- Huser, Opitz and Thibaud (2017):
  - $R > 0$, W Gaussian
  - asymptotic dependence/independence based on the distribution of R
- Huser and Wadsworth (2019):
  - R, W both Pareto-tailed, parameter δ controls relative heaviness
  - Smooth transistion between asymptotic dependence/independence $\longrightarrow$ the data decides

Inference is possible but only for up to about 30 locations.

Subasymptotic tail models, part III

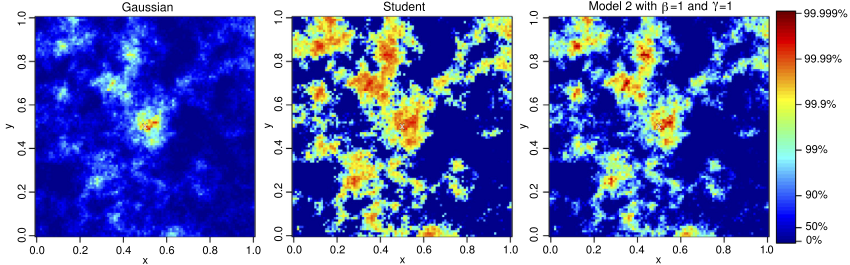
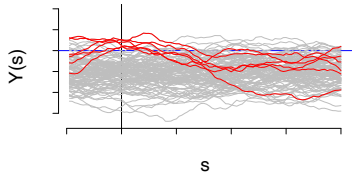


Fig. 1. Conditional simulations from an (asymptotically independent) Gaussian copula (left), an (asymptotically dependent) Student-t copula with 3 degrees of freedom (middle) and one of our new proposed Gaussian scale mixtures defined in (14) with $\beta = \gamma = 1$ leading to asymptotic independence (right), displayed on exponential margins. The underlying Gaussian field has exponential correlation function $\rho(\boldsymbol{s}_1, \boldsymbol{s}_2) = \exp(-\|\boldsymbol{s}_1 - \boldsymbol{s}_2\|/0.1)$. Simulation is conditional to the central grid cell at $(0.5, 0.5)$ being equal to the 99.99%-quantile. The conditional simulation of the scale variable $\tilde{R}$ is based on a Metropolis–Hastings algorithm. The three conditional simulations of the Gaussian component $\tilde{\boldsymbol{W}}$ use the same random seed.

$$F(r) = \begin{cases} 1 - \exp\left\{-\gamma(r^\beta - 1)/\beta\right\}, & \beta > 0, \\ 1 - r^{-\gamma}, & \beta = 0, \end{cases} \quad r \geq 1. \tag{14}$$

Spatial conditional extremes model

Spatial conditional extremes model arises when we consider $Y(s)|Y(s_0) > u$ (Heffernan and Tawn, 2004).



Spatial conditional extremes model, part II

Wadsworth and Tawn (2022) propose a model of the form

$$Y(s)|Y(s_0) > u \approx a_{s-s_0}(Y(s_0)) + b_{s-s_0}(Y(s_0))Z^0(s),$$

Process	$a_{s-s_0}(x)$	$b_{s-s_0}(x)$	$Z^0(s)$
Gaussian	$\rho(s-s_0)^2 x$	$1 + a_{s-s_0}(x)^{1/2}$	Gaussian
Brown–Resnick	x	1	Gaussian
t_ν	x	1	$t_{\nu+1}$ (transformed margins)
Inverted Brown–Resnick	$\ell_{s-s_0}(x)x$	$a_{s-s_0}(x)/(\log x)^{1/2}$	Independence (reverse Gumbel margins)
Huser and Wadsworth (2019) ($\lambda < 1$)	$\rho(s-s_0)^2 x$	$1 + a_{s-s_0}(x)^{1/2}$	Gaussian

- These models include asymptotic dependence and independence
- Likelihood-based inference is possible at 100s of sites
- Issue: A natural conditioning location usually doesn't exist

In summary

- Until recently, the availability of models for spatial extremes has been limited, and many of the existing models didn't match very well with what is observed in data
- Recent (theoretical) advances have substantially increased the pool of possible models
- Inference is still a challenge and a work in progress for many of these models
- However, the availability of good data is constantly growing, making it easier to infer complex structures from data

References

Heffernan, J. E., & Tawn, J. A. (2004). A conditional approach for multivariate extreme values (with discussion). *Journal of the Royal Statistical Society Series B: Statistical Methodology*, 66(3), 497-546.

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