Exercises and Lecture Notes
STK 4090, Spring 2020

Version 0.11, 18-Aug-2020

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Abstract

These are Exercises and Lecture Notes for the course Confidence, Likelihood, Probability, STK 4180 (Master level) or STK 9180 (PhD level), for the autumn semester 2020. I'll add on more exercises as the course progresses.

1. The probability transform

Some of the following facts are related to various operations for confidence distributions and confidence curves.

(a) Suppose X has a continuous and increasing cumulative distribution function F, i.e. $F(x) = \Pr\{X \le x\}$. Show that $U = F(X)$ is uniform on the unit interval. Any continously distributed random variable can hence be transformed to uniformity, via this *probability transform*.

(b) Show that also $U_2 = 1 - F(X)$ and $U_3 = |1 - 2\,F(X)|$ have uniform distributions.

(c) Simulate a million copies of $x_i \sim \mathrm{N}(0,1)$, and check the histogram of $\Gamma_1(x_i^2)$, where Γ_ν is the cumulative distribution function of a χ^2_ν. Comment on what you find.

(d) Suppose $\widehat\theta$ is an estimator for the real parameter θ, based on data y, with some continuous distribution function $K_\theta(x) = \Pr_\theta\{\widehat\theta \le x\}$; we are in particular assuming that the distribution of $\widehat\theta$ depends only on θ, not on other aspects of the underlying model employed. Consider the construction

$$C(\theta, y_{\rm obs}) = \Pr_\theta\{\widehat\theta \ge \widehat\theta_{\rm obs}\} = 1 - K_\theta(\widehat\theta_{\rm obs}),$$

a curve that can be computed and plotted post-data, where $\widehat\theta_{\rm obs} = \widehat\theta(y_{\rm obs})$ is the observed estimate. Show that it has the property that the random $C(\theta, Y)$ is uniformly distributed, for each fixed θ.

2. CD and cc for the normal standard deviation

Read Cunen and Hjort's *Confidence Curves for Dummies* (2020), a FocuStat Blog Post. Then do the details, regarding mathematics and implementation, for their introductory meant-to-be-simple example:

"Here's a simple example. You observe the data points 4.09, 6.37, 6.87, 7.86, 8.28, 13.13 from a normal distribution and wish to assess the underlying spread parameter, the famous standard deviation σ. We'll now introduce you to as many as two (2) curves: the confidence curve $\mathrm{cc}(\sigma)$ and the confidence distribution $C(\sigma)$. They're close cousins, actually, and it's not the case that both curves need to be displayed for each new statistical application."

(a) Here you might start with the classic fact concerning the empirical variance that $\widehat\sigma^2 \sim \sigma^2\chi^2_m/m$, where $m = n-1$, with n the sample size. Then deduce that

$$C(\sigma, y_{\rm obs}) = \Pr_\sigma\{\widehat\sigma \ge \widehat\sigma_{\rm obs}\} = 1 - \Gamma_m(m\widehat\sigma^2_{\rm obs}/\sigma^2).$$

Here $y_{\rm obs}$ represents the observed data, and $\widehat\sigma_{\rm obs}$ the observed point estimate. Show that $C(\sigma, Y) \sim \mathrm{unif}$, where Y represents a random data set $Y_1, \ldots, Y_n$, from the σ in question. In particular, the distribution of $C(\sigma.Y)$ does not depend on σ.

(b) Reproduce versions of Cunen and Hjort's Figures A and B, with the confidence curve $\mathrm{cc}(\sigma)$, the CD $C(\sigma)$, the median confidence estimate, etc.

(c) Compute also the *confidence density* $c(\sigma, y_{\rm obs})$ associated with the CD. Compute also its mode, say σ^*, and briefly assess its properties as an estimator of σ.

(d) A Bayesian approach to the same problem, i.d. finding a posterior distribution for σ, is to start with a prior $\pi(\sigma)$ and then compute $\pi(\sigma \,|\, y_{\rm obs}) \propto \pi(\sigma)g(\widehat\sigma, \sigma)$, where $g(\widehat\sigma, \sigma)$ is the likelihood, here the density function for $\widehat\sigma$ as a function of σ. When does such a Bayesian approach agree with the confidence density?

(e) Suppose there are two independent normal samples, with standard deviations σ_1 and σ_2. Construct a CD for $\rho = \sigma_1/\sigma_2$. Invent a second simple small dataset, to complement the first dataset given above, and then compute and display the confidence curve $\mathrm{cc}(\rho, \mathrm{data})$.

3. A skewed distribution on the unit interval

Consider the model $F(y, \theta) = y^\theta$ for observations on $[0, 1]$, where θ is an unknown positive parameter.

(a) Write down the log-likelihood function and find a formula for the maximum likelihood estimator $\widehat\theta$.

(a) Use theory of CLP, Chapter 2, to write down a normal approximation to the distribution of $\widehat\theta$.

(a) Consider the data set

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0.013 0.054 0.234 0.286 0.332 0.507 0.703 0.763 0.772 0.920
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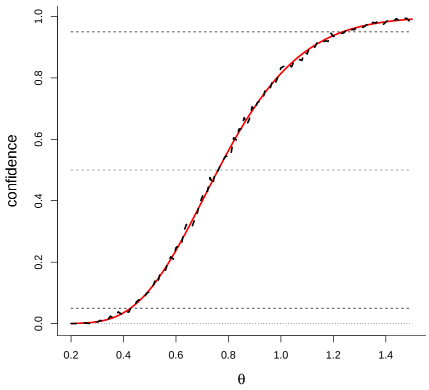
Estimate θ and compute the confidence distribution $C(\theta) = \Pr_\theta\{\widehat\theta \ge \widehat\theta_{\rm obs}\}$, along with the confidence curve $\mathrm{cc}(\theta) = |1 - 2\,C(\theta)|$, (i) using simulations, (ii) using exact probability calculus. Reproduce a version of Figure 0.1.

(a) Supplement these two curves with approximations based (i) on the normal approximation for $\widehat\theta$ and (ii) on the chi-squared approximation for the deviance.



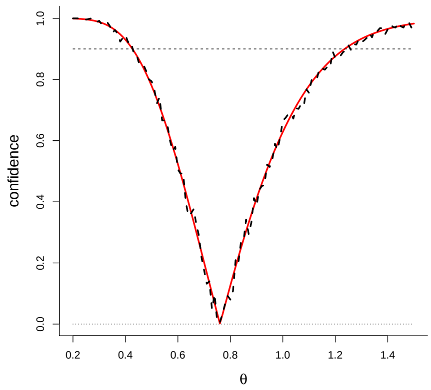


Figure 0.1: Left panel: confidence distribution $C(\theta)$, via simulations (black and wiggly curve) and via exact calculations (red and smooth curve); right panel: the two versions of the associated confidence curve $\mathrm{cc}(\theta)$.

References

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