Alternative environmental contours for structural reliability analysis

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Abstract

This paper presents alternative methods for constructing environmental contours for probabilistic structural reliability analysis of structures exposed to environmental forces such as wind and waves. For such structures, it is important to determine the environmental loads to apply in structural reliability calculations and structural design. The environmental contour concept is an effective, risk-based approach in establishing such design conditions. Traditionally, such contours are established by way of a Rosenblatt transformation from the environmental parameter space to a standard normal space, which introduces uncertainties and may lead to biased results. The proposed alternative approach, however, eliminates the need for such transformations and established environmental contours based on direct Monte Carlo sampling from the joint distribution of the relevant environmental parameters. In this paper, three alternative implementations of the proposed generic approach will be outlined.

Keywords: Environmental contours, Structural safety, Marine structures, Environmental loads and effects, Probabilistic structural design, Structural response, Structural reliability, First Order Reliability Method, Risk assessment

1. Introduction and background

Probabilistic structural reliability analysis is performed to ensure that a structure is able to withstand the required design loads. A realistic description of the environmental loads and structural response is a crucial prerequisite for structural reliability analysis of structures exposed to environmental forces. In principle, full long-term response analyses should be considered (Sagriolo et al. (2011)), but this is normally very time-consuming and computational intensive. The concept of environmental contours is an efficient method of estimating extreme conditions as basis for design (Winterstein et al. (1993), Haven and Winterstein (2009)), and is widely used in marine structural design (see e.g. Baarholm et al. (2010), Fontaine et al. (2013), Jonathan et al. (2011), Moan (2009), Ditlevsen (2002)). This approach is also recommended by DNV-GL (DNV (2014)). The main idea is to use the Rosenblatt transformation in order to transform the environmental variables into independent standard normally distributed variables and identify a sphere with desired radius in this transformed space. Environmental contours are then constructed by re-transforming the sphere back to the original space. This approach is closely related to the FORM-approximation (First Order Reliability Method), where the failure boundary in the transformed space is approximated by a hyperplane at the design point.

Transformations between the original space and the normal space will typically be non-linear. This makes the interpretation of the resulting contours less straightforward. Potential problems caused by this are discussed in detail in Huseby et al. (2013b). A brief introduction to probabilistic structural design and the traditional approach to environmental contours is also given in Huseby et al. (2013b).

In the present paper we will focus our attention on methods where the contours are constructed directly in the original space, utilizing Monte Carlo simulations of the joint environmental model. This yields a more straightforward interpretation of the contours. Another advantage is a more flexible framework for establishing environmental contours, which for example simplifies the inclusion of effects such as future projections of the wave climate related to climatic change (Vanem and Bitner-Gregersen (2012)). Other examples of applications of Monte Carlo methods in structural reliability analysis are presented in e.g. Naess et al. (2009), Naess et al. (2012), Zhang et al. (2010), Juncher Jensen et al. (2011).

Safety regulations and construction standards should ensure that structures are able to withstand the external loads and avoid structural failure. In probabilistic structural design, structural reliability analysis forms the basis for rule development. Such safety rules and standards should guide the individual design by restricting the allowable design space, but structural reliability analyses may also be used for individual designs. The main idea is to make sure that the reliability of the structure is sufficient, corresponding to a maximum allowable failure probability.
The reliability of marine structures implicit in safety rules is discussed in Bitter-Gregersen et al. (2002).

In the following, the proposed approach to environmental contours using direct Monte Carlo simulations will be briefly reviewed and three specific methods for identifying the contours are presented (Section 3). It is noted that the first method has been discussed previously in Huseby et al. (2013b), but it is included herein for completeness. Methods 2 and 3 are not previously published but Method 2 has been briefly presented in Huseby et al. (2013a).

In Section 5, the methods are illustrated by case studies where environmental contours for significant wave height and zero up-crossing wave period are calculated. Furthermore, environmental contours for significant wave height and wave steepness are presented, illustrating the need, in some cases, for a standardization of the variables when they are numerically very different. Section 6 discusses some important issues and compares the proposed alternative method with the traditional one and also presents a brief inter-comparison of the three methods. Finally, a summary is provided in Section 7. It is noted that in this paper the methods are illustrated in two dimensions only, but that, in general, they are easily extended to higher dimensions. The only requirement is that the joint distribution of stochastic input parameters, henceforth referred to as the environmental model, is possible to simulate from.

2. Environmental contours

Environmental contours are defined in various ways in the literature (Leira (2008), Moan et al. (2005), Haver (1987)). In this paper, however, the same understanding of environmental contours as used in Huseby et al. (2013b) will be held: Let \( \mathbf{X} \) be a vector of environmental variables with possible values in the set \( \mathcal{X} \subset \mathbb{R}^n \) and assume that the distribution of \( \mathbf{X} \) is absolute continuous with respect to the Lebesques measure in \( \mathbb{R}^n \). Moreover, let \( P_e \in (0, 0.5) \) be a given exceedence probability. The objective is to identify a convex set \( \mathcal{B} \subset \mathcal{X} \) such that for every supporting hyperplane\(^2\) \( \Pi \) of \( \mathcal{B} \), we have \( P[\mathbf{X} \in \Pi_+] = P_e \), where \( \Pi_+ \) denotes the halfspace bounded by the hyperplane \( \Pi \) and not containing \( \mathcal{B} \). We also introduce \( \Pi^- \) which denotes the halfspace complementary to \( \Pi_+ \). The resulting environmental contour is the boundary of the set \( \mathcal{B} \) and denoted \( \partial \mathcal{B} \). Whenever such a set \( \mathcal{B} \) can be found, safely designed structures can easily be identified.

Figure 1 illustrates how this can be done. For a given failure region \( \mathcal{F} \) we may check that the corresponding failure probability, \( P[\mathbf{X} \in \mathcal{F}] \), is bounded by the exceedence probability, \( P_e \), simply by verifying that \( \mathcal{F} \) is a convex set such that \( \mathcal{F} \cap \mathcal{B} = \emptyset \). If this is the case, it follows by standard convexity theory that there will always exist a hyperplane \( \Pi \) such that \( \Pi \) supports \( \mathcal{B} \) and such that \( \mathcal{F} \subseteq \Pi^+ \). Hence, obviously the failure probability \( P[\mathbf{X} \in \mathcal{F}] \) is less than or equal to \( P[\mathbf{X} \in \Pi^+] = P_e \), as stated.

In a real-life application the exact shape of the failure region is usually not determined in detail. Instead one investigates the states along the contour \( \partial \mathcal{B} \), or at least the most extreme ones, and verifies that the structure does not fail for any of these states. Given that this holds true, one typically assumes that the structure is safe for all the interior states in \( \mathcal{B} \) as well. Thus, one may conclude that the failure region does not intersect with \( \mathcal{B} \). It then only remains to argue that it is reasonable to assume that \( \mathcal{F} \) is convex. That is, if \( \mathbf{X}_1 \) and \( \mathbf{X}_2 \) are two arbitrary failure states, one must argue that \( \alpha \mathbf{X}_1 + (1 - \alpha) \mathbf{X}_2 \), where \( \alpha \in [0, 1] \), is a failure state as well. Even without exact knowledge about \( \mathcal{F} \), this may often be a reasonable assumption.

It is of interest to compare the direct simulation based approach to environmental contours constructed using the traditional approach, i.e., by using the Rosenblatt transformation. The Rosenblatt transformation is a transformation \( T : \mathcal{X} \rightarrow \mathbb{R}^n \) such that \( \mathbf{Y} = T(\mathbf{Y}) \) is a vector of \( n \) independent standard normally distributed variables. A contour for \( \mathbf{Y} \) with the desired exceedence probability, \( P_e \), can easily be constructed analytically, and the resulting contour in \( \mathcal{X} \) is obtained by using the inverse transformation of \( T \). As for the direct simulation approach, one proceeds by verifying that the structure does not fail for any of the states along the contour and thus concludes that the failure region does not intersect with region surrounded by the contour. However, in order to verify that the failure probability, \( P[\mathbf{X} \in \mathcal{F}] \), is bounded by the exceedence probability, \( P_e \), one must in this case argue that the corresponding transformed failure region \( T(\mathcal{F}) \) is convex. That is, if \( \mathbf{X}_1 \) and \( \mathbf{X}_2 \) are two arbitrary failure states, one must argue that \( T^{-1}(\alpha T(\mathbf{X}_1) + (1 - \alpha) T(\mathbf{X}_2)) \), where \( \alpha \in [0, 1] \), is a failure state as well. Since the transformation \( T \) depends on the joint probability distribution of \( \mathbf{X} \) and not just on the physical properties of the structure, it

\(^2\) A hyperplane \( \Pi \) is a supporting hyperplane of a convex set \( \mathcal{B} \) if \( \mathcal{B} \) is entirely contained in one of the two closed half-spaces determined by \( \Pi \) and \( \mathcal{B} \) has at least one boundary-point on \( \Pi \).
Then we have

Moreover, let the halfspace \( \tilde{\Pi} \) be chosen to be as large as possible. Thus, we let \( \Pi \) has the property mentioned above, we say that \( X \) admits a \( P_\varepsilon \)-contour. From the above definition we see that the construction of \( B \) is strongly linked to hyperplanes \( \Pi \) with the property that \( P[X \in \Pi^+] = P_\varepsilon \). We will refer to such hyperplanes as \( P_\varepsilon \)-exceedence hyperplanes, and we denote by \( \mathcal{P}(P_\varepsilon) \) the family of all \( P_\varepsilon \)-exceedence hyperplanes. The following basic result states that \( P_\varepsilon \)-exceedence hyperplanes actually exist.

**Proposition 2.1.** Let \( \Pi \) be an arbitrary hyperplane in \( \mathbb{R}^n \). Then there exists a \( P_\varepsilon \)-exceedence hyperplane \( \tilde{\Pi} \) which is parallel to \( \Pi \).

**Proof:** Let \( c \in \mathbb{R}^n \) be a orthogonal vector to \( \Pi \), and introduce the random variable \( Y = cX \). Since we have assumed that the distribution of \( X \) is absolute continuous with respect to the Lebesgue measure in \( \mathbb{R}^n \), it follows that the cumulative distribution function of \( Y \), \( F_Y(y) \) is a continuous function. Hence, there exists a number \( y_e \) such that \( P[Y > y_e] = (1 - F_Y(y_e)) = P_\varepsilon \). We then define the hyperplane \( \tilde{\Pi} \) as:

\[
\tilde{\Pi} = \{x : cx = y_e\}.
\]

Moreover, let the halfspace \( \tilde{\Pi}^+ \) be:

\[
\tilde{\Pi}^+ = \{x : cx > y_e\}.
\]

Then we have \( P[X \in \tilde{\Pi}^+] = P[cX > y_e] = P[Y > y_e] = P_\varepsilon \). Thus, \( \tilde{\Pi} \) is a \( P_\varepsilon \)-exceedence hyperplane. Moreover, \( \tilde{\Pi} \) has the same orthogonal vector as \( \Pi \). Hence, \( \tilde{\Pi} \) is parallel to \( \Pi \). \( \blacksquare \)

To ensure that \( B \) has the desired property, we must have:

\[
B \subseteq \Pi^-, \quad \text{for all } \Pi \in \mathcal{P}(P_\varepsilon).
\]

Hence, it follows that \( B \) should be chosen as a subset of the intersection of all the \( \Pi^- \)-sets. At the same time all the supporting hyperplanes of the set \( B \) should ideally be \( P_\varepsilon \)-exceedence hyperplanes. In order to achieve this \( B \) should be chosen to be as large as possible. Thus, we let \( B \) be uniquely defined as:

\[
B = \bigcap_{\Pi \in \mathcal{P}(P_\varepsilon)} \Pi^-.
\]

Note that all the halfspaces \( \Pi^- \in \mathcal{P}(P_\varepsilon) \) are convex sets. Hence, the definition of \( B \) given in (2) implies that the set \( B \) is indeed a **convex set** as well since it is an intersection of convex sets.

Unfortunately, however, this definition does not necessarily imply that **all** supporting hyperplanes of \( B \) are \( P_\varepsilon \)-exceedence hyperplanes. In the remaining part of the paper we consider the two-dimensional case where \( X = (X_1, X_2) \). In this case we have the following basic result:

**Proposition 2.2.** If \( \Pi \) is a supporting hyperplane of \( B \) such that \( B \subseteq \Pi^- \), then \( P[X \in \Pi^+] \geq P_\varepsilon \).

**Proof:** Assume conversely that \( P[X \in \Pi^+] < P_\varepsilon \). By Proposition 2.1, we can find a \( P_\varepsilon \)-exceedence hyperplane \( \Pi \), parallel to \( \Pi \). It follows that \( P[X \in \Pi^+] < P[X \in \Pi^+] \). However, this implies that \( B \subseteq \Pi^- \subset \Pi^- \), contradicting that \( \Pi \) is a supporting hyperplane of \( B \). \( \blacksquare \)

Note that it may happen that \( P[X \in \Pi^+] > P_\varepsilon \). In this case it is also possible to find a \( P_\varepsilon \)-exceedence hyperplane \( \Pi \), parallel to \( \Pi \). However, now \( \Pi \) is on the “wrong” side of \( \Pi \). Hence, we instead have \( B \subseteq \Pi^+ \subset \Pi^- \). Thus, in this case the existence of the \( P_\varepsilon \)-exceedence hyperplane \( \Pi \) does not prevent \( B \) from being a supporting hyperplane of \( B \).

In order to take a closer look at this, we choose an angle \( \theta \in [0, 2\pi) \), and let \( c = (\cos(\theta), \sin(\theta)) \). As in Proposition 2.1 we are interested in the probability distribution of \( Y = cX = X_1 \cos(\theta) + X_2 \sin(\theta) \). Since \( |c| = \cos^2(\theta) + \sin^2(\theta) = 1 \), it follows that \( Y = cX \) is simply the **scalar projection** of the vector \( X \) onto the vector \( c \). This is illustrated in Figure 2.

![Figure 2: The scalar projection Y of the random vector X onto the unit vector c.](image)

In the following it is convenient to introduce the function \( C(\theta) \) defined for \( \theta \in [0, 2\pi) \) as:

\[
C(\theta) = \inf\{C : P[X_1 \cos(\theta) + X_2 \sin(\theta) > C] = P_\varepsilon\}. \quad (3)
\]

Note that by Proposition 2.1 there exists at least one number \( C \) such that:

\[
P[X_1 \cos(\theta) + X_2 \sin(\theta) > C] = P_\varepsilon.
\]
Hence, the function $C(\theta)$ is well-defined. We also introduce:

\[ \Pi(\theta) = \{(x_1, x_2) : x_1 \cos(\theta) + x_2 \sin(\theta) = C(\theta) \} \]

\[ \Pi^+(\theta) = \{(x_1, x_2) : x_1 \cos(\theta) + x_2 \sin(\theta) > C(\theta) \}, \]

\[ \Pi^-(\theta) = \{(x_1, x_2) : x_1 \cos(\theta) + x_2 \sin(\theta) \leq C(\theta) \}. \]

By the definition of $C(\theta)$ it follows that $P[X \in \Pi^+(\theta)] = P_e$. Thus, for every $\theta \in [0, 2\pi)$, it follows that $\Pi(\theta)$ is a $P_e$-exceedence hyperplane. Thus, $\{\Pi(\theta) : \theta \in [0, 2\pi)\} \subseteq \mathcal{P}(P_e)$, and hence,

\[ \bigcap_{\theta \in [0, 2\pi]} \Pi^- \subseteq \bigcap_{\theta \in [0, 2\pi]} \Pi^-(\theta). \]

Moreover, let $\Pi$ be any arbitrary $P_e$-exceedence hyperplane. Then there exists an angle $\theta \in [0, 2\pi)$ and a constant $\tilde{C}$ such that:

\[ \Pi = \{(x_1, x_2) : x_1 \cos(\theta) + x_2 \sin(\theta) = \tilde{C} \} \]

By (3) it follows that $C(\theta) \leq \tilde{C}$. Thus, it follows that $\Pi^-(\theta) \subseteq \Pi^-$. Since this holds for any $P_e$-exceedence hyperplane, we have:

\[ \bigcap_{\theta \in [0, 2\pi]} \Pi^- \subseteq \bigcap_{\theta \in [0, 2\pi]} \Pi^-(\theta). \]

Combining this and recalling the definition of $\mathcal{B}$ given in (2), we conclude that $\mathcal{B}$ can be written as:

\[ \mathcal{B} = \bigcap_{\theta \in [0, 2\pi]} \Pi^- . \tag{4} \]

Hence, we can focus our attention on the family of $P_e$-exceedence hyperplanes indexed by $\theta$ denoted by $\mathcal{Q}(P_e)$ given by:

\[ \mathcal{Q}(P_e) = \{\Pi(\theta) : \theta \in [0, 2\pi)\}. \tag{5} \]

The following proposition describes the relation between properties of $\mathcal{B}$ and the family $\mathcal{Q}(P_e)$.

**Proposition 2.3.** Let $\mathcal{B}$ be defined as in (4) and let $\mathcal{Q}(P_e)$ be defined as in (5). Then the following two statements are equivalent:

1. Every supporting hyperplane of $\mathcal{B}$ is a $P_e$-exceedence hyperplane.
2. Every hyperplane $\Pi(\theta) \in \mathcal{Q}(P_e)$ is a supporting hyperplane of $\mathcal{B}$.

**Proof:** Assume first that Statement 1 is true, and let $\Pi(\theta) \in \mathcal{Q}(P_e)$. If $\Pi(\theta)$ is not a supporting hyperplane of $\mathcal{B}$, we can find a supporting hyperplane $\Pi$ parallel to $\Pi(\theta)$ such that:

\[ \Pi = \{(x_1, x_2) : x_1 \cos(\theta) + x_2 \sin(\theta) = \tilde{C} \}. \]

Since $\Pi$ is a supporting hyperplane of $\mathcal{B}$, it follows by Proposition 2.2 that $P[X \in \Pi^+] \geq P_e = P[X \in \Pi^+(\theta)]$.

Hence, we must have $\tilde{C} \leq C(\theta)$, and since obviously $\Pi \neq \Pi(\theta)$, the only possibility is that $\tilde{C} < C(\theta)$. However, by Statement 1 $\Pi$ is also a $P_e$-exceedence hyperplane. Thus, by (3) we must have $\tilde{C} \geq C(\theta)$ which is a contradiction. Hence, we conclude that $\Pi(\theta)$ must be a supporting hyperplane of $\mathcal{B}$, i.e., Statement 2 holds true.

Assume then that Statement 2 holds true, and let $\Pi$ be a supporting hyperplane of $\mathcal{B}$. Then there exists an angle $\theta \in [0, 2\pi)$ and a constant $\tilde{C}$ such that:

\[ \Pi = \{(x_1, x_2) : x_1 \cos(\theta) + x_2 \sin(\theta) = \tilde{C} \} \]

Using Proposition 2.2 again it follows that $P[X \in \Pi^+] \geq P_e$. If $\Pi$ is not a $P_e$-exceedence hyperplane, the only possibility is that $P[X \in \Pi^+] > P_e$. Hence, we must have $\tilde{C} < C(\theta)$. However, this implies that $\mathcal{B} \subseteq \Pi^- \setminus \Pi(\theta)$ which contradicts that $\Pi(\theta)$ is a supporting hyperplane of $\mathcal{B}$. Thus, we conclude that $\Pi$ must be a $P_e$-exceedence hyperplane, i.e., Statement 1 holds true.

According to Proposition 2.3 $\mathbf{X}$ admits a $P_e$-contour if and only if every hyperplane $\Pi(\theta) \in \mathcal{Q}(P_e)$ is a supporting hyperplane of $\mathcal{B}$. In order to understand how this condition may fail to hold, we choose an arbitrary hyperplane $\Pi(\theta) \in \mathcal{Q}(P_e)$. Moreover, we let $\theta_1 < \theta < \theta_2$ be such that $\theta_2 - \theta_1 < \pi$, and consider the hyperplanes $\Pi(\theta_1)$ and $\Pi(\theta_2)$. Finally, we let $(x_1, x_2)$ denote the point where $\Pi(\theta_1)$ and $\Pi(\theta_2)$ intersect. Now, two possible situations can occur. In Figure 3 we observe that $(x_1, x_2) \in \Pi^+(\theta)$ which contradicts that $\Pi(\theta)$ is a supporting hyperplane of $\mathcal{B}$. On the other hand, in Figure 4 $(x_1, x_2) \in \Pi^-(\theta)$ implying that $\Pi(\theta)$ cannot have any points in common with $\mathcal{B}$. Thus, in this case $\Pi(\theta)$ is not a supporting hyperplane of $\mathcal{B}$.

![Figure 3: Since $(x_1, x_2) \in \Pi^+(\theta)$, $\Pi(\theta)$ is potentially a supporting hyperplane of $\mathcal{B}$](image)

Based on the situation illustrated in Figure 3 a necessary and sufficient condition for the existence of a $P_e$-contour can be derived. That is, it is possible to show the following result:

**Theorem 2.4.** $\mathbf{X}$ admits a $P_e$-contour if and only if for any $\theta_1 < \theta < \theta_2$ such that $\theta_2 - \theta_1 < \pi$, the intersection of $\Pi(\theta_1)$ and $\Pi(\theta_2)$ belongs to $\Pi^+(\theta)$.

A more explicit condition can be found by considering the intersection point $(x_1, x_2)$ introduced above. Since this
point belongs to the two hyperplanes $\Pi(\theta_1)$ and $\Pi(\theta_2)$, $(x_1, x_2)$ can be found by solving the following linear equations:

\[
\begin{align*}
  x_1 \cos(\theta_1) + x_2 \sin(\theta_1) &= C(\theta_1), \\
  x_1 \cos(\theta_2) + x_2 \sin(\theta_2) &= C(\theta_2),
\end{align*}
\]

with the solution:

\[
\begin{align*}
  x_1 &= \frac{\sin(\theta_2)C(\theta_1) - \sin(\theta_1)C(\theta_2)}{\sin(\theta_2 - \theta_1)}, \\
  x_2 &= \frac{-\cos(\theta_2)C(\theta_1) + \cos(\theta_1)C(\theta_2)}{\sin(\theta_2 - \theta_1)}. 
\end{align*}
\]  

By the definition of $\Pi^+(\theta)$, the condition that $(x_1, x_2) \in \Pi^+(\theta)$ can be stated as:

\[
x_1 \cos(\theta) + x_2 \sin(\theta) > C(\theta). 
\]

Inserting the solution (7) into (8) it is easy to see that we have the following result:

**Theorem 2.5.** $X$ admits a $P_\epsilon$-contour if and only if for any $\theta_1 < \theta < \theta_2$ such that $\theta_2 - \theta_1 < \pi$, we have:

\[
\sin(\theta_2 - \theta)C(\theta_1) + \sin(\theta_1 - \theta)C(\theta_2) > \sin(\theta_2 - \theta_1)C(\theta).
\]

The condition given in Theorem 2.5 can be simplified somewhat by considering the special case where $\theta_1 = \theta - \delta$, $\theta_2 = \theta + \delta$, and $0 < \delta < \pi/2$. In this case the condition can be written as:

\[
\sin(\delta)C(\theta - \delta) + \sin(\delta)C(\theta + \delta) > \sin(2\delta)C(\theta). 
\]

Being a special case of the general condition given in Theorem 2.5 we obviously have that (9) is a necessary condition for the existence of a $P_\epsilon$-contour. It can be shown, however, that (9) is a sufficient condition as well. Moreover, by using that $\sin(2\delta) = 2\sin(\delta)\cos(\delta)$ and that $\sin(\delta) > 0$, we arrive at the following result:

**Theorem 2.6.** $X$ admits a $P_\epsilon$-contour if and only if for any $\theta \in [0, 2\pi)$ and $\delta \in (0, \pi/2)$, we have:

\[
\frac{1}{2}[C(\theta - \delta) + C(\theta + \delta)] > \cos(\delta)C(\theta). 
\]

We observe that if $C$ is convex in $[\theta - \delta, \theta + \delta]$, then the condition (10) holds for this particular $\theta$ and $\delta$. If $C$ is everywhere convex, then (10) will be satisfied for every $\theta \in [0, 2\pi)$ and $\delta > 0$. However, since $C$ typically will have the property that $\lim_{\theta \to 2\pi} C(\theta) = C(0)$, $C$ cannot be strictly convex everywhere. Thus, unless $C$ is constant, this function will have parts where it is concave as well. Intuitively, Theorem 2.6 states that $C$ in its concave parts should behave locally like the cosine function around zero. On the other hand, if $C$ has parts with a more dramatically concave shape, the condition (10) may not hold, and in such cases $X$ does not admit a $P_\epsilon$-contour.

We illustrate this result by presenting a simple example:

**Example 2.7.** Assume that $X_1$ and $X_2$ are independent normally distributed with $E(X_i) = \mu_i$ and $SD(X_i) = \sigma$, $i = 1, 2$. Then it easy to see that:

\[
\Pi(\theta) = \{(x_1, x_2) : x_1 \cos(\theta) + x_2 \sin(\theta) = C(\theta)\} = \{(x_1, x_2) : (x_1 - \mu_1)\cos(\theta) + (x_2 - \mu_2)\sin(\theta) = R\},
\]

where $R = \sigma q_\epsilon$ and $q_\epsilon$ denotes the $(1 - P_\epsilon)$-percentile of the standard normal distribution. That is, for $\theta \in [0, 2\pi)$, $\Pi(\theta)$ is a line orthogonal to the vector $(\cos(\theta), \sin(\theta))$ such that the distance from the point $(\mu_1, \mu_2)$ to $\Pi(\theta)$ is $R$. The function $C(\theta)$ is given by:

\[
C(\theta) = \mu_1 \cos(\theta) + \mu_2 \sin(\theta) + R. 
\]

We then calculate $C(\theta - \delta)$ and $C(\theta + \delta)$:

\[
C(\theta - \delta) = \mu_1 \cos(\delta) + \mu_2 \sin(\delta) + R,
\]

\[
C(\theta + \delta) = \mu_1 \cos(\delta) - \sin(\delta) + R.
\]

By combining these equations, we get that:

\[
\frac{1}{2}[C(\theta - \delta) + C(\theta + \delta)] = \mu_1 \cos(\delta) + \mu_2 \sin(\delta) + R = \cos(\delta)C(\theta) + (1 - \cos(\delta))R.
\]

Hence, since obviously $(1 - \cos(\delta))R > 0$, it follows that the condition given in Theorem 2.6 is satisfied.

The function $C(\theta)$ can be used to identify the boundary of the set $B$. In order to show this, we assume that $C(\theta)$ is differentiable. For a given angle $\theta \in [0, 2\pi)$ and a small number $\delta > 0$ we consider the intersection between the two $P_\epsilon$-exceedance hyperplanes $\Pi(\theta)$ and $\Pi(\theta + \delta)$. This point can be found by solving the following linear equations:

\[
\begin{align*}
  x_1 \cos(\theta) + x_2 \sin(\theta) &= C(\theta), \\
  x_1 \cos(\theta + \delta) + x_2 \sin(\theta + \delta) &= C(\theta + \delta),
\end{align*}
\]
with the solution similar to the one given in (7):
\[
\begin{align*}
x_1 &= \frac{\sin(\theta + \delta)C(\theta) - \sin(\theta)C(\theta + \delta)}{\sin(\delta)} \\
x_2 &= -\cos(\theta + \delta)C(\theta) + \cos(\theta)C(\theta + \delta)
\end{align*}
\] (13)
As \( \delta \to 0 \) the intersection point \((x_1, x_2)\) will converge to a point in \( \Pi(\theta) \) which we denote by \((x_1(\theta), x_2(\theta))\). Using l’Hôpital’s rule it is easy to see that \((x_1(\theta), x_2(\theta))\) is given by:
\[
\begin{pmatrix}
x_1(\theta) \\
x_2(\theta)
\end{pmatrix} = \begin{bmatrix}
C(\theta) & -C'(\theta) \\
C'(\theta) & C(\theta)
\end{bmatrix} \begin{bmatrix}
\cos(\theta) \\
\sin(\theta)
\end{bmatrix},
\] (14)
where \( C'(\theta) \) denotes the derivative of \( C(\theta) \).

We illustrate the use of this formula by calculating \( \partial \mathcal{B} \) for the binormal case:

**Example 2.8.** As in the previous example we assume that \( X_1 \) and \( X_2 \) are independent normally distributed with \( E(X_i) = \mu_i \) and \( SD(X_i) = \sigma_i \), \( i = 1, 2 \). Thus, the function \( C(\theta) \) is given by (11). Moreover, \( C'(\theta) \) becomes:
\[
C'(\theta) = -\mu_1 \sin(\theta) + \mu_2 \cos(\theta).
\] (15)
By inserting (11) and (15) into (14) we obtain:
\[
\begin{align*}
x_1(\theta) &= C(\theta)\cos(\theta) - C'(\theta)\sin(\theta) \\
&= \left[ \mu_1 \cos(\theta) + \mu_2 \sin(\theta) + R \right] \cos(\theta) \\
&= [-\mu_1 \sin(\theta) + \mu_2 \cos(\theta)] \sin(\theta) \\
&= \mu_1 + R \cos(\theta),
\end{align*}
\]
and:
\[
\begin{align*}
x_2(\theta) &= C'(\theta)\cos(\theta) + C(\theta)\sin(\theta) \\
&= [-\mu_1 \sin(\theta) + \mu_2 \cos(\theta)] \cos(\theta) \\
&= \mu_1 \cos(\theta) + \mu_2 \sin(\theta) + R \sin(\theta) \\
&= \mu_2 + R \sin(\theta),
\end{align*}
\]
Putting it all together we conclude that \( \partial \mathcal{B} \) can be written as:
\[
\begin{pmatrix}
x_1(\theta) \\
x_2(\theta)
\end{pmatrix} = \begin{bmatrix}
\mu_1 \\
\mu_2
\end{bmatrix} + \begin{bmatrix}
R & 0 \\
0 & R
\end{bmatrix} \begin{bmatrix}
\cos(\theta) \\
\sin(\theta)
\end{bmatrix}
\]
Thus, we conclude that \( \partial \mathcal{B} \) is a circle with radius \( R \) centered at \((\mu_1, \mu_2)\).

### 3. Estimating environmental contours using Monte Carlo simulation

In this section we show how to estimate environmental contours using Monte Carlo simulation. More specifically, we use Monte Carlo simulations in order to estimate the set \( \mathcal{B} \) introduced in the previous section, and then use its boundary \( \partial \mathcal{B} \) as the desired contour. For the motivation for this approach, reference is made to the discussions in Huseby et al. (2013b), where this was first introduced. As in the previous section a two-dimensional environmental space is assumed, and we let \( T \) and \( H \) denote the two environmental variables.

We start out by performing a Monte Carlo simulation on the joint environmental model (or alternatively on an empirical distribution) producing a total of \( n \) sample points:
\[
(T_1, H_1), \ldots, (T_n, H_n)
\] (16)

For a given angle \( \theta \in [0, 2\pi] \) we calculate the length of the projections of these points onto the unit vector \((\cos(\theta), \sin(\theta))\), i.e.:
\[
Y_i(\theta) = T_i \cos(\theta) + H_i \sin(\theta), \quad i = 1, \ldots, n
\] (17)
These lengths are then sorted in ascending order:
\[
Y_{(1)}(\theta) \leq Y_{(2)}(\theta) \leq \cdots \leq Y_{(n)}(\theta)
\] (18)
and we let \((\hat{T}_1, \hat{H}_1), \ldots, (\hat{T}_n, \hat{H}_n)\) denote the corresponding sample points, i.e.:
\[
Y_{(i)}(\theta) = T_1 \cos(\theta) + H_1 \sin(\theta), \quad i = 1, \ldots, n
\] (19)
Now, for a given exceedance probability \( P_e \), we calculate the number of samples \( k \) such that
\[
\frac{n - k}{n} \approx P_e.
\] (20)
Thus, \( k \approx n(1 - P_e) \). In fact, assuming that \( P_e \in (0, 1) \) is a rational number, we can always ensure that \( k = n(1 - P_e) \) by choosing \( n \) sufficiently large. Having identified \( k \), an unbiased estimate for \( \hat{C}(\theta) \) is:
\[
\hat{C}(\theta) = Y_{(k)}(\theta).
\] (21)
Based on the estimate of \( C(\theta) \) we also obtain unbiased estimates of the hyperplane \( \hat{\Pi}(\theta) \), as well as the two halfspaces \( \hat{\Pi}^+(\theta) \) and \( \hat{\Pi}^-(\theta) \) given by:
\[
\hat{\Pi}(\theta) = \left\{ (t, h) : t \cos(\theta) + h \sin(\theta) = \hat{C}(\theta) \right\},
\] (22)
\[
\hat{\Pi}^+(\theta) = \left\{ (t, h) : t \cos(\theta) + h \sin(\theta) > \hat{C}(\theta) \right\},
\] (23)
\[
\hat{\Pi}^-(\theta) = \left\{ (t, h) : t \cos(\theta) + h \sin(\theta) < \hat{C}(\theta) \right\}.
\] (24)
We observe that the number of sample points inside \( \hat{\Pi}^+(\theta) \) is \( n - k \). Hence, the probability \( P([T, H] \in \hat{\Pi}^+(\theta)) \) can be estimated to be \( (n - k)/n \approx P_e \). Thus, the line \( \hat{\Pi}(\theta) \) is indeed a reasonable estimate of a supporting hyperplane of the set \( \mathcal{B} \), orthogonal to the vector \((\cos(\theta), \sin(\theta))\).

By repeating this for all angles \( \theta \in [0, 2\pi] \), we get a family of supporting hyperplane estimates. By considering the intersection of the halfspaces defined by these supporting hyperplane estimates we can construct an estimate of the set \( \mathcal{B} \). In a real-life situation, however, we have to limit the process by considering only a finite number of angles,
\( \theta_1 < \cdots < \theta_r \). Typically, these angles will be chosen to be evenly distributed throughout the interval \([0, 2\pi)\). Thus, we get the following estimate of the set \( \mathcal{B} \):

\[
\mathcal{B} = \bigcap_{j=1}^{r} \Pi^{-}(\theta_j) \quad (25)
\]

Assuming that \((T, H)\) admits a \(P_r\)-contour, we should ideally have that all the hyperplanes \(\Pi(\theta_1), \ldots, \Pi(\theta_r)\) are supporting hyperplanes of \(\mathcal{B} \). However, due to e.g., sampling errors, this will not always be the case. By using the condition (10) given in Theorem 2.6 on \(\hat{C}(\theta)\) one can easily identify the hyperplanes which do not support \(\mathcal{B} \), and possibly eliminate these hyperplanes from the collection. Still in order to highlight problematic parts of the contour, we have chosen to keep all the hyperplanes. As a result there may be parts of the estimated contour which are slightly irregular.

Note that, while each supporting hyperplane is estimated without bias, the set \(\hat{\mathcal{B}} \) is a slightly biased estimate for the true set \(\mathcal{B} \). In fact assuming that we could generate an infinite number of samples so that each of the \(r\) hyperplanes could be estimated perfectly, we would still typically have that \(\mathcal{B} \subset \hat{\mathcal{B}} \). This bias can be reduced by increasing \(r\). However, this implies that we must increase the number of sorted projections defined in (18). If the number of sample points, \(n\), is large, the computational burden of producing such sorted projections can be quite substantial.

In the following, three algorithms for constructing environmental contours will be presented. Of these, the first method has previously been presented in Huseby et al. (2013b), but it is included herein for completeness. Methods 2 and 3 are new in this paper. Implementations of all three methods are available in Riscue\(^3\), which also supports Monte Carlo simulations from a wide range of joint environmental models.

### 3.1. Constructing the contour: Method 1

In the first method, the environmental contour is constructed based on the intersection points between neighbouring supporting hyperplanes. We denote these intersection points by \((t_1, h_1), \ldots, (t_r, h_r)\), where the \(j\)th intersection point, \((t_j, h_j)\), is the intersection between \(\Pi(\theta_j)\) and \(\Pi(\theta_{j+1})\), \(j = 1, \ldots, r\), and where we let \(\theta_{r+1} = \theta_1\). The points are found by solving the following equations

\[
\begin{align*}
t_j \cos(\theta_j) + h_j \sin(\theta_j) &= \hat{C}(\theta_j), \\
t_j \cos(\theta_{j+1}) + h_j \sin(\theta_{j+1}) &= \hat{C}(\theta_{j+1}).
\end{align*}
\quad (26)
\]

Assuming that \(\theta_{j+1} - \theta_j = \Delta\), \(j = 1, \ldots, r\), the solution similar to the one given in (7) becomes:

\[
\begin{align*}
t_j &= \frac{\sin(\theta_{j+1}) \hat{C}(\theta_j) - \sin(\theta_j) \hat{C}(\theta_{j+1})}{\sin(\Delta)}, \\
h_j &= -\frac{\cos(\theta_{j+1}) \hat{C}(\theta_j) + \cos(\theta_j) \hat{C}(\theta_{j+1})}{\sin(\Delta)},
\end{align*}
\quad (27)
\]

for \(j = 1, \ldots, r\). The resulting environmental contour is found by drawing straight line segments between these intersection points. That is, for \(j = 1, \ldots, r - 1\), we draw a line segment between the points \((t_j, h_j)\) and \((t_{j+1}, h_{j+1})\). Finally, one last line segment is drawn between \((t_r, h_r)\) and \((t_1, h_1)\).

If all the hyperplanes \(\Pi(\theta_1), \ldots, \Pi(\theta_r)\) are supporting hyperplanes of \(\mathcal{B} \), the contour constructed this way, is exactly the boundary of the set \(\hat{\mathcal{B}} \). If this is not the case, the resulting contour will contain points not in \(\mathcal{B} \). Typically, such points will produce small loops along the contour. By identifying the hyperplanes which do not support \(\mathcal{B} \) by using the condition (10), and eliminate these hyperplanes from the collection, these irregularities can easily be removed. However, these loops indicate uncertainties in the contour estimate, and thus provide some useful information. Hence, we usually include the full collection of hyperplanes.

As mentioned above the set \(\hat{\mathcal{B}} \) slightly overestimates the set \(\mathcal{B} \). Since the environmental contour constructed with this method represents the boundary of \(\hat{\mathcal{B}} \), this contour inherits the same bias. If all \(r\) hyperplanes are estimated without any error and support \(\mathcal{B} \), then the contour is the boundary of a polygon with \(r\) corners circumscribing \(\mathcal{B} \). The corners are the points, \((t_1, h_1), \ldots, (t_r, h_r)\), and these points will typically fall outside of the true set \(\mathcal{B} \). By using some sort of interpolation on the \(C\)-function, it is possible to increase the number of supporting hyperplanes without increasing the number of sorted projections (see (18)). However, this implies that the difference between the angles, \(\Delta\), becomes smaller. As a result both the numerators and the denominators in (27) become smaller as well possibly making the point estimates less stable numerically.

### 3.2. Constructing the contour: Method 2

The second method is based on the formula (14), but with the true value of the \(C\)-function and its derivative replaced by estimates. That is, we calculate an environmental contour by using the following formula for \(\theta \in [0, 2\pi)\):

\[
\begin{pmatrix}
  t(\theta) \\
  h(\theta)
\end{pmatrix} = \begin{pmatrix}
  \hat{C}(\theta) & -\hat{C}'(\theta) \\
  \hat{C}'(\theta) & \hat{C}(\theta)
\end{pmatrix} \begin{pmatrix}
  \cos(\theta) \\
  \sin(\theta)
\end{pmatrix},
\quad (28)
\]

In order to simplify the comparison between the contours obtained using Method 1 and 2, we initially assume that the same number of points are used in both cases. More specifically we assume that the new contour is obtained by connecting the points \((t(\theta_1), h(\theta_1)), \ldots, (t(\theta_r), h(\theta_r))\) in a similar way as in

\(^3\)The Riscue software may be used freely for non-commercial purposes and is available for download at http://www.riscue.org/.
Method 1. Assuming that all the \( r \) hyperplanes are estimated without any error and support \( B \), we have that \((t(\theta_j),h(\theta_j))\) is the point where \( \Pi(\theta_j) \) supports \( B \). From this it is easy to see that the new contour is the boundary of a polygon with \( r \) corners which is inscribed in \( B \). Thus, using Method 2 we also obtain a contour which is biased, but this time the polygon is smaller than \( B \). See Figure 5.

By using interpolation on \( \hat{C} \) and \( \hat{C}' \) a large number of points can easily be constructed. Fortunately, the formula (28) is less sensitive to numerical problems compared to the formula (27) used in Method 1. Hence, using Method 2 the bias problem can essentially be eliminated.

### 3.3. Constructing the contour: Method 3

The third method can be construed as a variant of method 2 described above, where a Fourier series expansion replaces interpolation in order to approximate the \( C \)-function and thereby improve the smoothness of the resulting environmental contours. Note that this expansion could in principle be used in combination with method 1 as well yielding yet another method. In the present paper, however, we have chosen not to investigate this further.

The \( C \)-function is arguably periodic, repeating itself with a period of \( 2\pi \). It is therefore possible to approximate the \( C \)-function by a Fourier expansion. Assuming that \( C(\theta) \) is known for all \( \theta \), the Fourier coefficients are computed using the standard formulae in (29).

\[
a_n = \frac{1}{\pi} \int_0^{2\pi} C(\theta) \cos(n\theta) d\theta \quad n \geq 0 \\
b_n = \frac{1}{\pi} \int_0^{2\pi} C(\theta) \sin(n\theta) d\theta \quad n \geq 1
\]  

(29)

The Fourier series of \( C(\theta) \) is then the infinite sum:

\[
C_F(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\theta) + b_n \sin(n\theta)]
\]

(30)

and for some suitable integer \( N \geq 1 \) the \( C \)-function can be approximated by the partial Fourier series \( C_{F,N}(\theta) \) in (31).

\[
C_{F,N}(\theta) = \frac{a_0}{2} + \sum_{n=1}^{N} [a_n \cos(n\theta) + b_n \sin(n\theta)]
\]

(31)

By replacing \( C \) with \( \hat{C} \) in (29) we obtain the following estimates of the Fourier coefficients:

\[
\hat{a}_n = \frac{1}{\pi} \int_0^{2\pi} \hat{C}(\theta) \cos(n\theta) d\theta \quad n \geq 0 \\
\hat{b}_n = \frac{1}{\pi} \int_0^{2\pi} \hat{C}(\theta) \sin(n\theta) d\theta \quad n \geq 1
\]

(32)

However, since \( \hat{C}(\theta) \) is only calculated for certain \( \theta \)-values corresponding to the selected supporting hyperplanes, the integrals in (32) must be estimated numerically. Finally, the estimated Fourier coefficients are inserted into the partial Fourier series (31), and thus we arrive at the following curve estimate:

\[
\hat{C}_{F,N}(\theta) = \frac{\hat{a}_0}{2} + \sum_{n=1}^{N} [\hat{a}_n \cos(n\theta) + \hat{b}_n \sin(n\theta)]
\]

(33)

Assuming that the Fourier expansion \( C_{F,N} \) is a close approximation to the true \( C \)-function, \( \hat{C}_{F,N} \) will typically be a satisfactory estimate of the true \( C \)-function as well. Moreover, \( \hat{C}_{F,N} \) is defined for all \( \theta \in [0,2\pi) \), and it is straightforward to compute the derivative \( \hat{C}'_{F,N} \). The contours can then be calculated using (28) where \( \hat{C}(\theta) \) and \( \hat{C}'(\theta) \) are replaced by \( \hat{C}_{F,N}(\theta) \) and \( \hat{C}'_{F,N}(\theta) \) respectively. Depending on the number of Fourier terms needed to get a good fit, this method works well even with a moderate number of supporting hyperplanes and it is therefore often more efficient than the previous method.

### 4. Standardization methods

In some cases there may be big differences in the variations between the two environmental variables. See e.g., Section 5.3. This may have a significant effect on the precision of the contour estimates. In order to explain this, we recall that the function \( C(\theta) \) is estimated by \( \hat{C}(\theta) \) only for a finite set of angles \( \theta_1 < \cdots < \theta_r \) evenly distributed throughout the interval \([0,2\pi)\). This works fine as long as the variation in the \( C \)-function is moderate. In cases where variation in the first environmental variable differs significantly from the variation in the second environmental variable, however, the resulting \( C \)-function will vary much more. Similar effects can be seen if there is a strong correlation between the environmental variables. In order to estimate the \( C \)-function in such cases, it would in fact be better to distribute the angles so that the angles are closer together in areas where there are significant changes in the \( C \)-function, and more spread out in areas where the \( C \)-function is more stable. Finding the optimal configuration of these angles, however, is not easy. An alternative
and much easier strategy is to use some kind of standardization of the environmental variables. Two alternative standardization methods, univariate standardization and bivariate standardization, are presented in the subsections below. Both standardization methods standardize the simulated values of the environmental parameters before calculating the contours. When the contours have been calculated, according to either of the methods 1 - 3, they are transformed back into the original scale. Note that both the standardization methods use linear transformations. Thus, these methods will not encounter the same issues as with the Rosenblatt transformation used in the traditional approach.

4.1. Univariate standardization

The univariate standardization method transforms the environmental variables individually without taking into account any correlation between the variables. We start out by generating a set of simulated values \((X_1, Y_1), \ldots, (X_n, Y_n)\). These are transformed into \((S_1, T_1), \ldots, (S_n, T_n)\) using the following linear transformation \((i = 1, \ldots, n)\):

\[
S_i = \frac{X_i - m_X}{\sigma_X},
\]

\[
T_i = \frac{Y_i - m_Y}{\sigma_Y},
\]

where \(m_X\) and \(\sigma_X\) are respectively the sample mean and standard deviation of the \(X_i\)'s, while \(m_Y\) and \(\sigma_Y\) are respectively the sample mean and standard deviation of the \(Y_i\)'s. It is then very easy to verify that the resulting sample means of the \(S_i\)'s and the \(T_i\)'s are zero, while the sample standard deviations are 1.

A contour is then calculated as explained in Section 3 yielding points \((s_1, t_1), \ldots, (s_k, t_k)\) along the contours for \((S, T)\), where \(k\) is the chosen number of points. These points are then transformed back into the original \((X, Y)\)-space, where the resulting points are denoted \((x_1, y_1), \ldots, (x_k, y_k)\). These points are calculated as follows \((j = 1, \ldots, k)\):

\[
x_j = \sigma_X s_j + m_X,
\]

\[
y_j = \sigma_Y t_j + m_Y.
\]

4.2. Bivariate standardization

The bivariate standardization method differs from the univariate standardization method in that it also adjusts for strong correlations between \(X\) and \(Y\). Again, we start out by generating a set of simulated values \((X_1, Y_1), \ldots, (X_n, Y_n)\). These are transformed into \((S_1, T_1), \ldots, (S_n, T_n)\) using the following linear transformation \((i = 1, \ldots, n)\):

\[
S_i = \frac{X_i - m_X}{\sigma_X},
\]

\[
T_i = \left[\frac{Y_i - m_Y}{\sigma_Y} - \rho_{X,Y} \frac{X_i - m_X}{\sigma_X}\right] \sqrt{1 - \rho_{X,Y}^2},
\]

where as before \(m_X\) and \(\sigma_X\) denote respectively the sample mean and standard deviation of the \(X_i\)'s, while \(m_Y\) and \(\sigma_Y\) denote respectively the sample mean and standard deviation of the \(Y_i\)'s. Finally, \(\rho_{X,Y}\) denotes the sample correlation between the \(X_i\)'s and \(Y_i\)'s. Again it is very easy to verify that the resulting sample means of the \(S_i\)'s and the \(T_i\)'s are zero, while the sample standard deviations are 1. Moreover, in this case the sample correlation between the \(S_i\)'s and \(T_i\)'s becomes zero as well. Note that if \(\rho_{X,Y} = 0\), the bivariate standardization formulas reduces to the same formulas used for univariate standardization.

As for univariate standardization we then calculate \(k\) points \((s_1, t_1), \ldots, (s_k, t_k)\) along the contours for \((S, T)\). These points are then transformed back into the original \((X, Y)\)-space to the points \((x_1, y_1), \ldots, (x_k, y_k)\), where \((j = 1, \ldots, k)\):

\[
x_j = \sigma_X s_j + m_X,
\]

\[
y_j = \sigma_Y \rho_{X,Y} s_j + \sqrt{1 - \rho_{X,Y}^2} t_j + m_Y.
\]

5. Case studies

In order to illustrate the alternative methods for constructing environmental contours, some case studies where specific environmental models have been adopted are presented below. Note that the proposed approach to environmental contour lines are not dependent on the type of joint model employed. In fact, since the contours can be estimated based on empirical distributions and bootstrap methods, it is not even necessary to have a specific parametric model. Still, since the interest is often in extreme events, corresponding to observations far into the tail of the distribution, empirical data usually contain limited information of such extreme events. Hence, using a parametric model is recommended in order to extrapolate and estimate tail behaviour. Nevertheless, this is not a requirement for the proposed method for environmental contours.

The case studies presented below, are based on a conditional model for significant wave height and wave period, but the approach proposed in this paper would work equally well for other environmental models. Other bivariate modelling approaches for significant wave height and wave period are discussed in e.g., Ferreira and Guedes Soares (2002), Jonathan et al. (2010), Der Kiureghian and Liu (1986), and Repko et al. (2004) contains an overview of different methods for bivariate modelling. Multivariate modelling of metocean parameters by means of kernel density models are discussed in Athanassoulis and Belibassakis (2002). Ditlevsen (2002) presented some conditional contour curves for the significant wave height and wave period, given different wind pressure conditions.

5.1. Joint environmental model for significant wave height and wave period

The joint model for significant wave height and wave period applied in the case studies, is based on the conditional
modelling approach described in (38), where the marginal distribution of significant wave height \((H_s)\) is assumed to follow a 3-parameter Weibull distribution, while a conditional log-normal distribution is used for the wave period (zero up-crossing period, \(T_z\)) (Bitner-Gregersen et al. (1995)). This is the same model that was used in Vanem and Bitner-Gregersen (2012) and Huseby et al. (2013b).

\[
f_{H_s,T_z}(h,t) = f_{H_s}(h)f_{T_z|H_s}(t|h) \quad (38)
\]

The 3-parameter Weibull distribution is parametrized by a location parameter \(\gamma\), a scale parameter \(\alpha\) and a shape parameter \(\beta\), as follows

\[
f_{H_s}(h) = \frac{\beta}{\alpha} \left(\frac{h - \gamma}{\alpha}\right)^{\beta-1} e^{-\left[(h-\gamma)/\alpha\right]^\beta}, \quad h \geq \gamma \quad (39)
\]

The parameters of the log-normal distribution for wave period is dependent on \(H_s\) as shown in (40). The fitted model parameters are shown in Table 1 (Vanem and Bitner-Gregersen (2012)).

\[
\mu_t(h) = E[\ln T_z|H_s = h] = a_1 + a_2 h^{a_3}
\]

\[
\sigma_t(h) = sd[\ln T_z|H_s = h] = b_1 + b_2 e^{b_3 h} \quad (40)
\]

Table 1: Fitted parameters for the joint environmental model

<table>
<thead>
<tr>
<th>(H_s)</th>
<th>(\alpha)</th>
<th>(\beta)</th>
<th>(\gamma)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.776</td>
<td>1.471</td>
<td>0.8888</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(T_z)</th>
<th>(i = 1)</th>
<th>(i = 2)</th>
<th>(i = 3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a_i)</td>
<td>0.1000</td>
<td>1.489</td>
<td>0.1901</td>
</tr>
<tr>
<td>(b_i)</td>
<td>0.0400</td>
<td>0.1748</td>
<td>-0.2243</td>
</tr>
</tbody>
</table>

5.2. Environmental contours for significant wave height and wave period

Using the joint distribution of \(H_s\) and \(T_z\) as specified above, environmental contours for 1-, 10- and 25-year environmental conditions were obtained by all the three suggested methods. In order to see the effect of the sample size, two separate simulations were carried out, using 1 million and 5 million samples respectively. The resulting contours are illustrated in Figures 6 and 7. The contours shown in 6 are constructed using the set of simulated data with 1 million samples, while the contours shown in 7 are constructed using the set of simulated data with 5 million samples. We observe that the contours obtained using the larger data set are noticeably smoother compared to those obtained from the smaller set. This effect is also pointed out in Huseby et al. (2013b). Increasing the number of samples even further would give even better results, but would increase computational time. Instead of increasing the sample size, it is also possible to apply the importance sampling technique presented in Huseby et al. (2014). By using this technique the precision is improved while the sample size can be kept at a moderate level. Finally, we note that for this particular case the smoothest results are obtained using Method 2.

Table 2: Points along the environmental contours; 5 million samples

<table>
<thead>
<tr>
<th>Angle (\theta) (°)</th>
<th>((T_z, H_s))-points along the 25-year contour</th>
</tr>
</thead>
<tbody>
<tr>
<td>Method 1</td>
<td>Method 2</td>
</tr>
<tr>
<td>0</td>
<td>(13.73, 12.53)</td>
</tr>
<tr>
<td>6</td>
<td>(13.74, 14.17)</td>
</tr>
<tr>
<td>12</td>
<td>(13.76, 13.91)</td>
</tr>
<tr>
<td>18</td>
<td>(13.67, 14.33)</td>
</tr>
<tr>
<td>42</td>
<td>(13.47, 14.65)</td>
</tr>
<tr>
<td>60</td>
<td>(13.28, 14.77)</td>
</tr>
<tr>
<td>72</td>
<td>(13.00, 14.86)</td>
</tr>
<tr>
<td>78</td>
<td>(13.42, 14.73)</td>
</tr>
<tr>
<td>84</td>
<td>(13.19, 14.77)</td>
</tr>
<tr>
<td>90</td>
<td>(13.17, 14.78)</td>
</tr>
<tr>
<td>96</td>
<td>(13.36, 14.78)</td>
</tr>
<tr>
<td>102</td>
<td>(13.24, 14.77)</td>
</tr>
<tr>
<td>108</td>
<td>(13.03, 14.72)</td>
</tr>
<tr>
<td>120</td>
<td>(13.26, 14.79)</td>
</tr>
<tr>
<td>132</td>
<td>(12.77, 14.57)</td>
</tr>
<tr>
<td>138</td>
<td>(12.34, 14.18)</td>
</tr>
<tr>
<td>144</td>
<td>(10.80, 12.47)</td>
</tr>
<tr>
<td>150</td>
<td>(3.51, 2.44)</td>
</tr>
</tbody>
</table>

The \((T_z, H_s)\) values of selected points along the contours for the 25-year extremes are presented for all three methods in Table 2, based on the simulations of 5 million samples. Only values corresponding to angles up to 150° are shown since higher angles correspond to low values of \((T_z, H_s)\) which are obviously not interesting from a structural reliability point of view. These values could then be used in the response analyses of any structural problem. The 25-year extreme condition adopted in Vanem and Bitner-Gregersen (2012) for response calculations was \((T_z, H_s) = (13.20, 14.62)\) and it can be seen from Table 2 that this is in reasonable agreement to the values found with the three methods of the alternative approach to environmental contours, with similar conditions found along the 25-year contour for angles between 30 - 120°. However, slightly more extreme conditions are identified with the new approach, with design conditions of \((T_z, H_s) = (13.00, 14.86), (T_z, H_s) = (13.21, 14.80)\) and
5.3. Environmental contours for significant wave height and wave steepness

In the example above, the variations of both parameters were in the same order of magnitude. In other cases, however, the variations can be very different. In such cases there is a need for some sort of a standardization of the variables before the contours are calculated. In order to illustrate this, we consider construction of environmental contours for a joint distribution of significant wave height and significant wave steepness. Wave steepness is defined as the ratio of wave height over wave length and the significant wave steepness is a measure of the average wave steepness of a sea state (see also (42) below).

Rogue waves, sometimes also referred to as monster waves, freak waves, killer waves, abnormal waves or simply extreme waves, are waves with individual wave height exceeding twice the significant wave height\(^4\), and may occur in sea states that are not identified as extreme by the environmental contours of significant wave height and mean zero-crossing period. Such waves, even if not occurring in an extreme sea state, may impose extreme loads on marine structures, and have the potential to be dimensioning for the structural design. Hence, the contribution from freak waves in rogue sea states with less than extreme significant wave heights should also be considered. In order to take such contributions into account, there is a need to identify the extreme rogue sea states associated with the required return period. One way of identifying such extreme rogue sea states is to establish environmental contours for significant wave height and significant wave steepness. In the following, such environmental contours will be calculated based on the same parametric models for significant wave height and wave period as used in the previous case study (see (38)-(40) and Table 1).

Some sea states are more prone to generate rogue waves than others. Typically, rogue waves tend to occur in steep sea states characterized by a narrow frequency spectrum and a narrow directional spectrum (Bitner-Gregersen and Toffoli (2012a)). The Benjamin-Feir index (BFI) is a measure related to the ratio of steepness or wave non-linearity to randomness or width of the wave spectrum (Janssen (2003)), defined as in (41). \(\varepsilon\) is a measure of non-linearity or wave steepness, \(\delta \omega\) denotes the width of the frequency spectrum and \(\omega_p\) is the spectral peak frequency.

\(^4\)Note that alternative definitions of rogue waves exist

Figure 6: Environmental contours obtained from 1 million samples; Method 1 (left), Method 2 (centre) and Method 3 (right). Wave period (Tz) in seconds along the abscissa and Significant Wave height (Hs) in meters along the ordinate. Contours corresponding to 1-, 10- and 25-year return periods are shown in green, blue and purple colours, respectively.

Figure 7: Environmental contours obtained from 5 million samples; Method 1 (left), Method 2 (centre) and Method 3 (right).
It has been demonstrated that the value of the Benjamin-Feir index can be used to quantify the probability of rogue waves in a sea state (Shemer (2010)). However, the Benjamin-Feir index can only be calculated based on spectral information of the waves. Thus, one would in principle need spectral wave information in order to identify rogue sea states, but most often the information available to the engineer is on the form of scatter diagrams with significant wave height and zero-up-crossing or mean wave period without detailed information about the wave spectrum. An extension of the Benjamin-Feir index that includes directional effects in the spectral bandwidth is proposed in (Mori et al. (2011)).

Nevertheless, it is known that the wave steepness, defined as the ratio of wave height to wave length, is important for the generation of rogue waves, with waves becoming increasingly unstable for higher wave steepness. In deep waters, the maximum wave steepness corresponding to the breaking wave limit is approximately $S_{\text{max}} = \frac{H_s}{L} \approx \frac{1}{12}$ (with $L$ denoting the wave length and $H_b$ the breaking wave limit).

The significant wave steepness, $S_s$, is a measure of the average wave steepness of a sea state and is estimated by the expression in (42), where $H_s$ is significant wave height, $T_z$ is the mean zero-up-crossing period and $g$ is the gravitational constant (DNV (2014)). A similar expression in terms of the peak period, $T_p$, can also be used.

$$S_s = \frac{2\pi H_s}{g T_z^2} \approx \frac{H_s}{1.567 T_z^2}$$

The Benjamin-Feir index is plotted against wave steepness in Bitner-Gregersen and Toffoli (2012b) and it is demonstrated that they are highly correlated. Hence, measures of the wave steepness of a sea state can be used to indicate the probability of occurrence of rogue waves, with increasing probability of occurrence for higher significant wave steepness.

Having simulated a number of pairs of $(H_s, T_z)$, it is straightforward to transform this into a set of $(H_s, S_s)$ and to construct environmental contours on the resulting data. It is observed that typical values for the significant wave steepness are much lower than typical values for significant wave height. For a set of simulations carried out, the significant wave height values range between less than 1 meter and almost 18 meters whereas the corresponding range of significant wave steepness values were from 0.006 to 0.13. Hence, the variation in significant wave height is more than 100 times as large as the variation in significant wave steepness. Fortunately this issue is easily handled using the standardization methods introduced in Section 4.

$$BFI = \frac{\epsilon \sqrt{2}}{\delta \omega / \omega_p} \quad (41)$$

Figure 8 shows environmental contours for significant wave height and wave steepness with and without standardization. Actual data for wave steepness were not available, but the wave steepness has been calculated according to (42) based on simulations from the joint model fitted to $H_s$ and $T_z$ used in the previous case study. The environmental contours are calculated by using Method 1 based on 3 million samples and correspond to 1-year, 10-year and 25-year return periods, respectively. Note that the plots shown in Figure 8 are scaled so that the variations appear to be about the same for both variables. Still the axis for wave steepness, i.e., the horizontal axis, ranges only from 0 to 0.16, while the axis for significant wave height ranges from 0 to 20.

We observe that the contours obtained without standardization do not fit the scatter at all, while the contours obtained with standardization are much better. While this effect becomes even more dramatic due to the scaling of the plot, it is obviously much to gain here by using standardization. For this example we used bivariate standardization. However, the correlation between the two variables is not very strong, so we would have gotten almost identical contours using univariate standardization.

6. Discussion

6.1. Differences between the traditional and the alternative approach

The most important advantage of the proposed approach to establishing environmental contours is that there is no need for (non-linear) transformations of the environmental variables. As has been demonstrated in Huseby et al. (2013b), such transformations may result in contour lines where the supporting hyperplanes fail to have the desired properties in the original parameter space. This may introduce bias and uncertainties in the estimated design sea states. In the example presented in Huseby et al. (2013b), the difference between the 25-year sea states estimated by the traditional approach and the proposed alternative approach was roughly 4 cm in significant wave height. This difference increased somewhat in the example presented in this paper, due to the smaller sample size (5 million compared to 10 million), which indicates that the number of samples influences the accuracy of the method. This is obviously as expected and is due to the decreasing Monte Carlo variance for larger sample sizes, but the Monte Carlo variance can be reduced quite effectively by the importance sampling technique outlined in Huseby et al. (2014). Nevertheless, differences of this order of magnitude are presumably negligible in most practical applications.

However, for other joint distributions, the differences between the traditional and the alternative approach to environmental contours can be quite significant, as demonstrated in the comparison study presented in Vanem and Bitner-Gregersen (2014). This quite clearly illustrates the
Figure 8: Environmental contours for $H_s$ and $S_s$: without standardization (left) and bivariate standardization (right). Significant wave steepness along the abscissa and significant wave height along the ordinate.

differences between the estimated contours, and highlights the fundamentally different interpretation of the contours. One striking difference is that whilst the points along the traditional contours typically correspond to realistic sea states, they lack a clear probabilistic interpretation in the original space. For the alternative contours, however, the points along the contours have a well-defined probabilistic interpretation with respect to the exceedence hyperplanes, and should not necessarily be construed as physical sea states. Further comparison studies that considers actual limit state functions and different environmental models are recommended to gain further insight into the differences between the two methods and on the implications this will have on practical engineering applications.

Another advantage of the proposed method is that it does not require a standard parametric model for the environmental parameters. This makes it simpler to include e.g., the effect of long term trends in the wave climate. In Vanem and Bitner-Gregersen (2012), such a trend was included by modifying the marginal distribution for significant wave height. Alternative trends in the wave climate have been estimated in e.g. Vanem et al. (2012a,b, 2014, 2012c); Vanem and Walker (2012); for a more complete overview, see Vanem (2013). Even though the modified distribution in Vanem and Bitner-Gregersen (2012) seemed to be a good approximation, it would have been more satisfactory if the environmental contours could have been established without using such an approximation. Indeed, with the proposed method for calculating the environmental contours, a modified joint model is not needed. Instead, Monte Carlo simulations could just as easily have been performed directly on the initial model and on the trend contribution separately and then samples from each of them could be combined to produce samples from the resulting distribution of sea state parameters with a climatic trend. Hence, the proposed approach allows for more flexibility in how such trends can be modelled and taken into account.

However, in structural reliability applications, a very high reliability is often required, corresponding to environmental contours far out in the tails of the environmental distributions. In such situations, empirical distributions may not be expected to contain sufficient information of the tail-behaviour of the distributions and it may still be useful to fit a parametric model in order to extrapolate further out in the tail. Hence, in most practical situations, it will make sense to use a parametric model for the environmental parameters. Hence, both the traditional approach and the alternative approach proposed herein suffer from the same uncertainties with regards to fitting a distribution to the available data. It has not been investigated how the amount of data influences these uncertainties and if both approaches are equally affected. Still, with the new approach it is straightforward to include long term trend effects as an additive parametric or non-parametric distribution so that the resulting distribution might be a mixture of two contributions. The new approach would manage to estimate environmental contours for such mixed distributions without any difficulties. Furthermore, contours based on dual models due to combined wind sea and swell or due to other combined wave systems could be established.

### 6.2. Computational performance

The traditional approach to environmental contours is generally very computationally effective, and it is implemented in various structural analysis software packages. Even though the computational performance of the alternative approach has not been directly compared to this, computational performance is not a serious issue. In fact Monte Carlo simulations of the joint environmental model, even for number of sample sizes up to 10 million, only take a matter of seconds. The computational time for calculating the contours themselves obviously increases with increasing number of simulations and with increasing angular resolution. However, even with 10 million simulations, contours based on 60 different angles are produced within a couple of minutes. This is usually more than fast enough.
for most engineering applications. Thus, computational efficiency should be no major obstacle for the alternative approach.

Furthermore, the computational performance of the proposed methods can be significantly improved by adopting the importance sampling scheme suggested in Huseby et al. (2014).

6.3. Contours accounting for rogue waves

It has been demonstrated how to construct environmental contours for rogue sea states by drawing contour plots in the \((H_s, S_s)\)-space, which are quite straightforward to establish with the proposed approach. In this way, extreme sea states prone to generate rogue waves can be identified that are not necessarily extreme in the \((H_s, T_s)\)-space. By doing this and combining with further analyses of short-term characteristics of the sea states, it is possible to more directly design to account for rogue waves. It is recommended to use these as alternative design sea states in marine structural design. Due to the fact that the range of values for \(H_s\) and \(S_s\) are of different order of magnitude, it may be necessary to perform some type of standardization of the simulated samples before calculating the contours. Two methods for doing this have been suggested, which both are demonstrated to work well.

6.4. Method 1 vs. Method 2 and Method 3

Three practical methods of implementing the proposed approach to environmental contour lines have been suggested in this paper, all made available in the Riscue software. Method 3 can be construed as a variant of Method 2, where Fourier series are used to approximate the \(C\)-function. Hence, the most fundamental difference is between Method 1 and Methods 2 and 3. In the following, first a comparison between Methods 1 and 2 will be presented, and then some consideration of method 3 will be given.

In general, environmental contour lines based on the second method are somewhat smoother than those calculated by the first method, even though there are still a few artefacts in areas where the curvature of the contour lines are large. However, these are less noticeable compared to the results obtained from the first method.

On the other hand, the contour lines obtained from the first method may be more intuitive and somewhat easier to interpret. The points along the curve obtained from Method 1 are intersection points between two supporting hyperplanes, meaning that the different contour segments between the intersection points are segments of such supporting hyperplanes. Hence, if a structural limit state function lies entirely outside of any one of these contour segments, i.e. any of the supporting hyperplanes, one can be sure that this structure will have at least the required reliability. For the second method, the intersection points lie at the intersection with the supporting hyperplane and the line defined by the angle \(\theta\) and the contour segments between these points cannot as easily be identified as part of any supporting hyperplane.

As can be concluded from the brief discussion above, there are reasons to prefer either implementation method of the proposed approach. However, it is noted that the three methods will converge and eventually become identical as the number of angles increases. Thus, for a sufficiently large number of \(\theta_s\), the methods will be practically identical and give the same environmental contour lines. The environmental contours obtained by the two first methods from one and the same simulation of 5 million samples from the environmental model in the case study referred to above are illustrated in Figure 9. The figures show contours for 1-, 10-, and 25-year extreme conditions with 30, 60 and 180 angles respectively. As can be seen from these plots, the second method reduces the irregularities along the contours, but less so for increasing angular resolutions. It is also noted that the difference between the methods will diminish for increasing sample size.

6.5. Determining the number of Fourier terms to include in Method 3

The third method is a variation of Method 2 where the \(C\)-function is approximated with a partial Fourier series. This could result in smoother environmental contours, but the results are highly sensitive to the number of Fourier terms included in the partial series, i.e. on a specified integer \(N\) (FORM terms in the Form Plot Settings in Riscue corresponds to \(N + 1\), e.g. specifying FORM terms = 1 term corresponds to \(N = 0\) and includes only the constant term \(\frac{C}{2}\) of the Fourier series). To illustrate this sensitivity, the six environmental contours in Figure 10 result from varying the number of Fourier terms to include, i.e. with 2, 5, 10, 20, 40 and 50 terms (corresponding to \(N = 1, 4, 9, 19, 39\) and 49) respectively, keeping all other settings unchanged. The same simulated data as in Figure 9 were used, with the same reliability levels and with 60 supporting hyperplanes and 60 curve points for all the contours. It can be seen that too low or too high number of Fourier components yield poor results, but for \(N\) in the range between 10 and 20 the contours seem to be reasonable in this example. With too few Fourier terms, the contours fail to capture the general form of the contours but with too many Fourier terms the contours become increasingly noisy.

It is noted that specifying a higher number of supporting hyperplanes to use allows a higher number of Fourier terms in the partial Fourier series. The most suitable number of terms to use in the Fourier expansion may not be immediately easy to determine, as elaborated on below, and would in most cases require some trial and error. However, if an adequate number of terms is specified, the resulting approximation of the \(C\)-function will be good and also the derivative will be accurately estimated.

There is an intimate relation between the number of supporting hyperplanes used and the number of Fourier terms that can be estimated with a reasonable precision.
Figure 9: Comparison of environmental contours obtained from the same simulated data for Method 1 (top) and Method 2 (bottom); using 30 different angles (left), 60 angles (centre) and 180 angles (right)

Figure 10: Environmental contours obtained with Method 3 (Fourier method) for different number of Fourier terms; N = 2 (top left), N = 5 (top centre), N = 10 (top right), N = 20 (bottom left), N = 40 (bottom centre) and N = 50 (bottom right)
In a Fourier expansion, the lower order terms reflect the long-wave oscillations whereas the higher order terms reflect higher frequency oscillations. It will be possible to estimate the long-wave cycles even with few supporting hyperplanes (i.e., few estimated values of the $C$-function), but little information can be extracted regarding the high frequency components. Hence, with few supporting hyperplanes there will not be sufficient information to accurately estimate the higher order coefficients in the Fourier series. Therefore, the more supporting hyperplanes one specifies, the more Fourier coefficients can be determined with reasonable precision. For example, if 10 Fourier terms are to be included, the highest order terms correspond to oscillations of length $2\pi/10 = 36^\circ$. In order to identify such short wave cycles, there should in principle be at least one observation for each of the four phases of the wave (crest, zero-crossing, trough, zero-crossing). In other words, for the case of 10 Fourier terms, there should be an estimated value of the $C$-function for each multiple of $36/4 = 9^\circ$. This corresponds to $360/9 = 40$ estimated $C(\theta)$-values or 40 supporting hyperplanes.

At the same time, the short-wave signals are prone to disappear in noise, so the corresponding Fourier-coefficients are typically difficult to estimate with any precision regardless of the interval of estimated $C$-values. Including a large number of terms in the Fourier series would correspond to trying to fit a function to fluctuations in the $C$-values that might in reality be nothing more than a result of Monte Carlo uncertainty. It is therefore difficult to give generic advice on the optimum number of supporting hyperplanes and Fourier-terms. The best choice would depend on the actual distributions for which the environmental contours are to be estimated. Nevertheless, as a rule of thumb, it may be said that the number of supporting hyperplanes should ideally be at least four times the number of Fourier-terms, even though some trial and error may be needed to determine the optimum settings on a case-by-case basis. Typically, reasonable settings may be number of supporting hyperplanes in the order of 60 - 120 and including 10 - 20 Fourier-terms.

7. Summary and conclusions

This paper has presented an alternative approach for establishing environmental contour lines for use in structural reliability analysis, e.g., for extreme response calculations in extreme sea conditions. The approach is demonstrated to give very similar results compared to the traditional approach in some well-behaved cases, even though it arguably has some fundamental differences. One is that the resulting contours need to be interpreted differently. In fact, the new approach could be interpreted directly in the original space in much the same way as the traditional approach should be interpreted in the transformed space: points along the contours correspond to hyperplanes which act as boundaries of the failure regions. For the traditional approach, however, the points along the contours in the original space do not have an obvious interpretation. Furthermore, the proposed approach is more flexible in that it does not require a joint parametric model for the environmental parameters. This makes it easier to include effects such as those related to including long-term trends and future projections of the wave climate into the calculations.

The computational performance is reasonable and should be satisfactory in most practical applications. Three practical methods for implementing the alternative approach have been presented, all with some pros and cons. However, for increased sample sizes and angular resolutions, the three methods will converge. It is also demonstrated how environmental contour plots can easily be established for rogue sea states, identifying extreme combinations of significant wave height and significant wave steepness with the desired return periods. Since these parameters have very different ranges of values, there is a need for standardization of the parameter values, and two specific standardization techniques have been introduced. Both seem to work well in correcting the numerical problems that may arise.

Hence, the approach presented in this paper is offered as an alternative approach to construct environmental contours based on direct Monte Carlo simulations.

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