

UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: STK4400 — Risk and reliability analysis

Day of examination:

Examination hours: –

This problem set consists of 5 pages.

Appendices: None

Permitted aids: Calculator

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

Let $\mathbf{X}$ be a vector of environmental variables with sample space $\mathcal{X} \subseteq \mathbb{R}^n$. Furthermore let $P_e \in (0, 0.5)$ be a given exceedance probability. We shall in this exercise look into how one can identify a convex set $\mathcal{B} \subset \mathcal{X}$ such that for all hyperplanes Π that are tangents to $\mathcal{B}$, we have $P[\mathbf{X} \in \Pi^+] = P_e$, where Π^+ denotes the halfspace that is bounded by the hyperplane Π and not containing $\mathcal{B}$. We also introduce Π^- denoting the halfspace that is complementary to Π^+ , such that $\mathcal{B} \subseteq \Pi^-$. The boundary of $\mathcal{B}$ is denoted by $\partial\mathcal{B}$ and is what is called an *environmental contour*.

(a) Explain how such an environmental contour can be used to check whether a certain mechanical structure is safe.

We assume in the rest of the exercise that $\mathbf{X} = (T, H)$. For $\theta \in [0, 2\pi)$ we define the random variable $Y(\theta) = T \cos(\theta) + H \sin(\theta)$. We also introduce the function $C(\theta)$ defined for $\theta \in [0, 2\pi)$:

$$C(\theta) = \inf\{C : P[T \cos(\theta) + H \sin(\theta) > C] = P_e\}.$$

(b) What property does $C(\theta)$ have related to the probability distribution of $Y(\theta)$?

We also introduce:

$$\begin{aligned}\Pi(\theta) &= \{(t, h) : t \cos(\theta) + h \sin(\theta) = C(\theta)\}, \\ \Pi^+(\theta) &= \{(t, h) : t \cos(\theta) + h \sin(\theta) > C(\theta)\}, \\ \Pi^-(\theta) &= \{(t, h) : t \cos(\theta) + h \sin(\theta) \leq C(\theta)\}.\end{aligned}$$

Then it can be shown that the set $\mathcal{B}$ can be written as:

$$\mathcal{B} = \bigcap_{\theta \in [0, 2\pi)} \Pi(\theta)^-.$$

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We consider the intersection point $(t(\theta_1, \theta_2), h(\theta_1, \theta_2))$ between the hyperplanes $\Pi(\theta_1)$ and $\Pi(\theta_2)$.

(c) Show the first of the two following relations:

$$t(\theta_1, \theta_2) = \frac{\sin(\theta_2)C(\theta_1) - \sin(\theta_1)C(\theta_2)}{\sin(\theta_2 - \theta_1)}$$

$$h(\theta_1, \theta_2) = \frac{-\cos(\theta_2)C(\theta_1) + \cos(\theta_1)C(\theta_2)}{\sin(\theta_2 - \theta_1)}$$

Then let $(t(\theta), h(\theta)) = \lim_{\delta \rightarrow 0^+} (t(\theta, \theta + \delta), h(\theta, \theta + \delta))$.

(d) Show that:

$$\begin{pmatrix} t(\theta) \\ h(\theta) \end{pmatrix} = \begin{bmatrix} C(\theta) & -C'(\theta) \\ C'(\theta) & C(\theta) \end{bmatrix} \cdot \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix},$$

where $C'(\theta)$ denotes the derivative of $C(\theta)$, and explain in a short way how this expression can be used to identify $\partial\mathcal{B}$.

(e) What sort of contour is $\partial\mathcal{B}$ if $C(\theta) = R$ for all $\theta \in [0, 2\pi)$? Give an explanation for your answer.

Assume we have carried through a Monte Carlo simulation based on the joint probability distribution of (T, H) , and that we have generated n vectors:

$$(T_1, H_1), \dots, (T_n, H_n) \tag{1}$$

For a given $\theta \in [0, 2\pi)$ we then calculate:

$$Y_i(\theta) = T_i \cos(\theta) + H_i \sin(\theta), \quad i = 1, \dots, n \tag{2}$$

(f) Explain how we can use $Y_1(\theta), \dots, Y_n(\theta)$ to estimate $C(\theta)$.

Problem 2

In this problem we consider the oil production from a field consisting of two reservoirs that share a processing facility with a constant process capacity $K > 0$. We then introduce:

$f_i(t)$ = The potential production rate at time $t \geq 0$ from reservoir i , $i = 1, 2$.

$q_i(t)$ = The actual production rate at time $t \geq 0$ from reservoir i , $i = 1, 2$.

$Q_i(t)$ = The cumulative production at time $t \geq 0$ from reservoir i , $i = 1, 2$.

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The *ultimately recoverable volumes* from each of the two reservoirs are V , and we assume that:

$$f_i(t) = f_i(Q_i(t)) = K \cdot \left[1 - \frac{Q_i(t)}{V}\right]^\alpha, \quad t \geq 0, \quad i = 1, 2,$$

$$q_i(t) = b_i(t) \cdot f_i(Q_i(t)), \quad t \geq 0, \quad i = 1, 2,$$

where $\alpha > 0$ and the *production strategy* $\mathbf{b}(t) = (b_1(t), b_2(t))$ is chosen so that for all $t \geq 0$ we have:

$$0 \leq b_i(t) \leq 1, \quad i = 1, 2,$$

and

$$b_1(t)f_1(Q_1(t)) + b_2(t)f_2(Q_2(t)) \leq K$$

(a) Define an *admissible* production strategy.

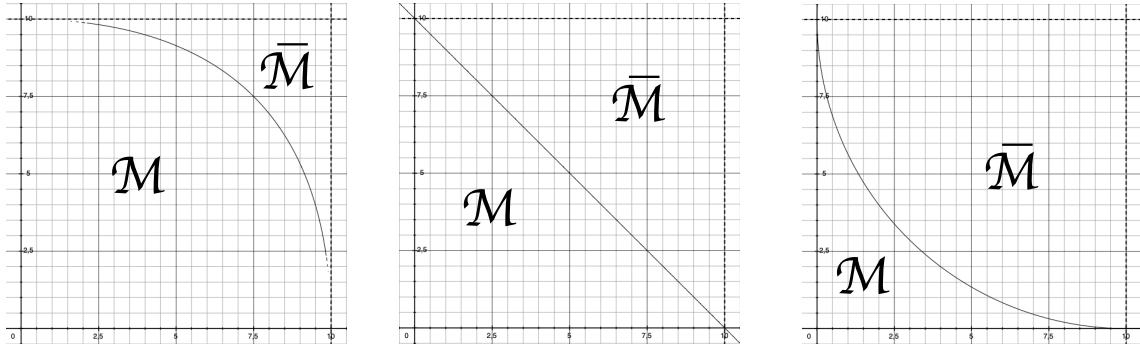


Figure 1: The sets $\mathcal{M}$ and $\bar{\mathcal{M}}$ when α is 0.5, 1.0 and 2.0 respectively.

We introduce the vector $\mathbf{Q} = (Q_1, Q_2)$ and the set $\mathcal{Q} = [0, V] \times [0, V]$ and let:

$$\mathcal{M} = \{\mathbf{Q} \in \mathcal{Q} : f_1(Q_1) + f_2(Q_2) \geq K\},$$

$$\bar{\mathcal{M}} = \{\mathbf{Q} \in \mathcal{Q} : f_1(Q_1) + f_2(Q_2) < K\}.$$

(b) Figure 1 shows the sets $\mathcal{M}$ and $\bar{\mathcal{M}}$ for the cases $\alpha = 0.5, 1.0, 2.0$. Assume that we want to produce the two reservoirs so that the total production is kept at the level K as long as possible. Describe the optimal production strategy in these three cases. No detailed calculations are required.

(c) What is the relation between the shapes of the sets $\mathcal{M}$ and $\bar{\mathcal{M}}$ and the shapes of the functions f_1 and f_2 ?

Problem 3

We consider a random variable X with cumulative distribution function $F_X(x) = P(X \leq x)$ and survival function $S_X(x) = P(X > x) = 1 - F_X(x)$. The α -level *value-at-risk* associated

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with X is defined as follows:

$$\text{V@R}_\alpha(X) = S_X^{-1}(\alpha) = \inf\{x : P(X > x) \leq \alpha\}.$$

It can be shown that if S_X is *strictly decreasing*, then:

$$P(X > x_0) \leq \alpha \leq P(X \geq x_0) \quad \Rightarrow \quad \text{V@R}_\alpha(X) = x_0.$$

(a) Assume that S_X is *strictly decreasing*. Show that:

$$P(X > x_0) = \alpha \quad \Rightarrow \quad \text{V@R}_\alpha(X) = x_0.$$

(b) Let ϕ be a strictly increasing continuous function. Show that:

$$\text{V@R}_\alpha[\phi(X)] = \phi(S_X^{-1}(\alpha)) = \phi(\text{V@R}_\alpha[X])$$

A cedent has m non-negative *absolutely continuously* distributed risks:

$$X_1, \dots, X_m$$

The risks cannot be *bundled* and covered by a single contract. Instead separate reinsurance contracts must be purchased for each risk. All risks are reinsured using *insurance layer* contracts:

$$R_i(X_i) = \begin{cases} 0 & X_i < a_i \\ X_i - a_i & a_i \leq X_i \leq b_i \\ b_i - a_i & X_i > b_i \end{cases} \quad i = 1, \dots, m.$$

where $0 < a_i < b_i$ are the *contract parameters* for the i th risk, $i = 1, \dots, m$. As a result, the *retained risks* covered by the cedent are:

$$I_i(X_i) = \begin{cases} X_i & X_i < a_i \\ a_i & a_i \leq X_i \leq b_i \\ X_i - (b_i - a_i) & X_i > b_i \end{cases} \quad i = 1, \dots, m.$$

(c) Draw a sketch of the function $I_i(X_i)$, and discuss briefly how the cedent's risk exposure varies with the contract parameters.

The price paid by the cedent for the i th contract is assumed to be of the form:

$$(1 + \theta)E[R_i(X_i)], \quad i = 1, \dots, m.$$

where $\theta > 0$ represents the risk premiums charged by the reinsurance company for handling the risks. The total risk covered by the cedent is then:

$$\Psi = \sum_{i=1}^m I_i(X_i) + (1 + \theta) \sum_{i=1}^m E[R_i(X_i)].$$

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(d) Show that the resulting total α -level value-at-risk, i.e., the upper α -percentile of the distribution of Ψ , is given by:

$$\text{V@R}_\alpha(\Psi) = \text{V@R}_\alpha\left(\sum_{i=1}^m I_i(X_i)\right) + (1 + \theta) \sum_{i=1}^m E[R_i(X_i)].$$

(e) Assume that the contract parameters $a_1, b_1, \dots, a_m, b_m$ are chosen so that:

$$P\left(\sum_{i=1}^m I_i(X_i) \geq \sum_{i=1}^m a_i\right) \geq \alpha \quad \text{and} \quad P\left(\sum_{i=1}^m I_i(X_i) > \sum_{i=1}^m a_i\right) \leq \alpha.$$

Show that this implies that:

$$\text{V@R}_\alpha(\Psi) = \sum_{i=1}^m a_i + (1 + \theta) \sum_{i=1}^m E[R_i(X_i)]$$

(f) Assume that the contract parameters $a_1, b_1, \dots, a_m, b_m$ are chosen so that:

$$P\left(\sum_{i=1}^m I_i(X_i) > \sum_{i=1}^m a_i\right) = \alpha.$$

Show that this implies that:

$$\text{V@R}_\alpha(\Psi) = \sum_{i=1}^m a_i + (1 + \theta) \sum_{i=1}^m E[R_i(X_i)]$$

(g) Assume that we have determined the optimal contract parameters $a_1, \dots, a_m$. Explain briefly why the optimal contract parameters $b_1, \dots, b_m$ can be found by solving the following optimization problem:

$$\begin{aligned} &\text{Minimize: } \sum_{i=1}^m E[R_i(X_i)] \\ &\text{Subject to: } P\left(\sum_{i=1}^m I_i(X_i) > \sum_{i=1}^m a_i\right) = \alpha. \end{aligned}$$

with respect to $b_1, \dots, b_m$.