

Multi-reservoir production optimization

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Multi-reservoir production optimization

- Part 1. Optimization in the deterministic case:

Huseby and Haavardsson (2008) *A Framework For Multi-reservoir Production Optimization*, Sections 1-4

- Part 2. Optimization in the stochastic case:

Huseby and Sødal (2012) *Sequential optimization of oil production from multiple reservoirs under uncertainty*



Actual production restricted by processing capacity

Consider the oil production from a field consisting of n reservoirs that share a processing facility with a constant process capacity $K > 0$.

$\mathbf{Q}(t) = (Q_1(t), \dots, Q_n(t)) =$ Cumulative production functions

$\mathbf{f}(t) = (f_1(t), \dots, f_n(t)) =$ Pot. production rate (PPR) functions

We assume that the *ultimately recoverable volumes* from the n reservoirs are respectively $V_1, \dots, V_n$, and that:

$$0 \leq Q_i(t) \leq V_i, \quad i = 1, \dots, n.$$

Moreover, we assume that:

$$f_i(t) = f_i(Q_i(t)), \quad t \geq 0, \quad i = 1, \dots, n.$$

Typically, f_i will be a decreasing function of Q_i , $i = 1, \dots, n$.



Actual production restricted by processing capacity

If the sum of the potential production rates exceeds the capacity K of the processing facility, i.e.,

$$\sum_{i=1}^n f_i(t) > K$$

the production needs to be *choked*.

$\mathbf{q}(t) = (q_1(t), \dots, q_n(t)) =$ Actual production rates after choking

$q(t) = \sum_{i=1}^n q_i(t) =$ Total production rate at time t

$Q(t) = \sum_{i=1}^n Q_i(t) =$ Total cumulative production at time t



Production strategy

A *production strategy* is defined for all $t \geq 0$:

$$\mathbf{b} = \mathbf{b}(t) = (b_1(t), \dots, b_n(t)),$$

where $b_i(t)$ represents the *choke factor*, i.e., the fraction of the potential production rate of the i th reservoir that is actually produced at time t , $i = 1, \dots, n$.

The *actual production rates* from the reservoirs after the production is choked are given by:

$$q_i(t) = \frac{dQ_i(t)}{dt} = b_i(t)f_i(Q_i(t)), \quad i = 1, \dots, n.$$



Valid production strategies

To satisfy the physical constraints of the reservoirs and the process facility, we require that:

$$0 \leq b_i(t) \leq 1, \quad i = 1, \dots, n, \quad t \geq 0.$$

$$\sum_{i=1}^n b_i(t) f_i(Q_i(t)) \leq K.$$

$\mathcal{B}$ denotes the class of production strategies that satisfy these physical constraints. We refer to production strategies $\mathbf{b} \in \mathcal{B}$ as *valid production strategies*.



Admissible production strategies

An *admissible production strategy* is defined as a valid production strategy $\mathbf{b}$ where the total production rate $q(t)$ satisfies the following constraint for all $t \geq 0$:

$$q(t) = \sum_{i=1}^n b_i(t) f_i(Q_i(t)) = \min\{K, \sum_{i=1}^n f_i(Q_i(t))\}.$$

$\mathcal{B}' \subseteq \mathcal{B}$ denotes the class of admissible strategies.

If $\mathbf{b} \in \mathcal{B}'$ and $T_K(\mathbf{b}) = \sup\{t \geq 0 : \sum_{i=1}^n f_i(Q_i(t)) \geq K\}$ is the corresponding plateau length, then:

$$\begin{aligned} q(t) &= K, & 0 \leq t \leq T_K(\mathbf{b}). \\ q(t) &= \sum_{i=1}^n f_i(Q_i(t)), & t > T_K(\mathbf{b}). \end{aligned}$$



Objective functions

An *objective function* is a mapping $\phi : \mathcal{B} \rightarrow \mathbb{R}$ such that if $\mathbf{b}^1, \mathbf{b}^2 \in \mathcal{B}$, we prefer $\mathbf{b}^2$ to $\mathbf{b}^1$ if $\phi(\mathbf{b}^2) \geq \phi(\mathbf{b}^1)$.

An *optimal production strategy* with respect to ϕ is a production strategy $\mathbf{b}^{opt} \in \mathcal{B}$ such that $\phi(\mathbf{b}^{opt}) \geq \phi(\mathbf{b})$ for all $\mathbf{b} \in \mathcal{B}$.



Monotone objective functions

Definition

An objective function ϕ is said to be monotone if for any pair of production strategies $\mathbf{b}^1, \mathbf{b}^2 \in \mathcal{B}$ such that $\mathbf{Q}(t, \mathbf{b}^1) \leq \mathbf{Q}(t, \mathbf{b}^2)$ for all $t \geq 0$ we have $\phi(\mathbf{b}^1) \leq \phi(\mathbf{b}^2)$.

Proposition

Let ϕ be a monotone objective function, and let $\mathbf{b}^1, \mathbf{b}^2 \in \mathcal{B}$ be such that $\mathbf{b}^1(t) \leq \mathbf{b}^2(t)$ for all $t \geq 0$. Then $\phi(\mathbf{b}^1) \leq \phi(\mathbf{b}^2)$.

Proposition

Let ϕ be a monotone objective function, and let $\mathbf{b} \in \mathcal{B}$. Then there exists $\mathbf{b}' \in \mathcal{B}'$ such that $\phi(\mathbf{b}') \geq \phi(\mathbf{b})$.

Symmetric objective functions

Definition

An objective function ϕ is said to be symmetric if it depends on a production strategy $\mathbf{b}$ only through the total production rate function $q(\cdot, \mathbf{b})$ (or equivalently through $Q(\cdot, \mathbf{b})$).

Proposition

Let ϕ be a symmetric objective function. Then ϕ is monotone if and only if for any pair of production strategies, $\mathbf{b}^1$ and $\mathbf{b}^2$ such that $Q(t, \mathbf{b}^1) \leq Q(t, \mathbf{b}^2)$ for all $t \geq 0$, we have $\phi(\mathbf{b}^1) \leq \phi(\mathbf{b}^2)$.



Symmetric objective functions (cont.)

Proposition

Let ϕ be a symmetric objective function, and let $\mathbf{b} \in \mathcal{B}'$. Then $\phi(\mathbf{b})$ is uniquely determined by $\mathbf{Q}(T_K(\mathbf{b}))$. Thus, we may write $\phi(\mathbf{b}) = \phi(\mathbf{Q}(T_K(\mathbf{b})))$.

Since ϕ is assumed to be symmetric, it depends on $\mathbf{b}$ only through q . Furthermore, since $\mathbf{b} \in \mathcal{B}'$, we know that $q(t) = K$ whenever $0 \leq t \leq T_K(\mathbf{b})$. This implies that:

$$Q(T_K(\mathbf{b})) = \sum_{i=1}^n Q_i(T_K(\mathbf{b})) = K T_K(\mathbf{b}).$$

Hence, the plateau length $T_K(\mathbf{b})$ can be recovered from $\mathbf{Q}(T_K(\mathbf{b}))$ as:

$$T_K(\mathbf{b}) = K^{-1} \sum_{i=1}^n Q_i(T_K(\mathbf{b})).$$



Symmetric objective functions (cont.)

If $t > T_K(\mathbf{b})$, it follows since $\mathbf{b} \in \mathcal{B}'$ that:

$$q(t) = \sum_{i=1}^n q_i(t) = \sum_{i=1}^n f_i(Q_i(t))$$

By the Picard-Lindelöf's theorem $q_i(t)$ is uniquely determined for all $t > T_K(\mathbf{b})$ by its respective differential equation along with the boundary condition given by the value $Q_i(T_K(\mathbf{b}))$, $i = 1, \dots, n$.

Thus, $q(t)$ is uniquely determined by $\mathbf{Q}(T_K(\mathbf{b}))$ for all $t \geq 0$, and hence so is ϕ



Symmetric objective functions (cont.)

As an example we consider the following objective function:

$$\phi(\mathbf{b}) = \int_0^\infty I\{q(u) \geq C\} q(u) e^{-Ru} du,$$
$$0 \leq C \leq K, \quad R \geq 0,$$

where R is a discount factor, and C is a threshold value reflecting the minimum acceptable production rate. We refer to this objective function as a *truncated discounted production* objective function.

This objective function is both *monotone* and *symmetric*. Hence, for any admissible production strategy, $\mathbf{b}$, $\phi(\mathbf{b})$ is uniquely determined by $\mathbf{Q}(T_K(\mathbf{b}))$.



Optimizing production strategies

In order to study the optimization problem further we introduce the following sets:

$$\mathcal{Q} = [0, V_1] \times \cdots \times [0, V_n],$$

$$\mathcal{M} = \{\mathbf{Q} \in \mathcal{Q} : \sum_{i=1}^n f_i(Q_i) \geq K\},$$

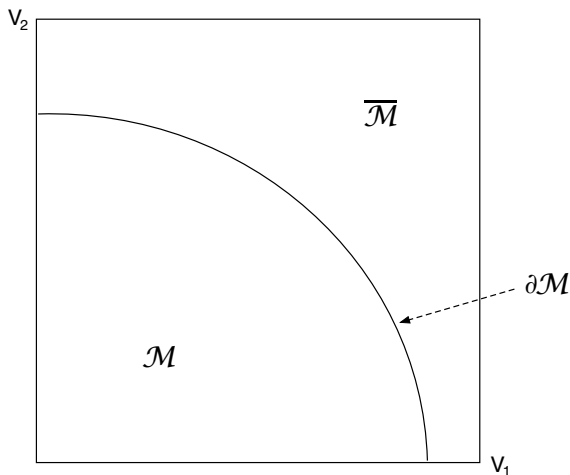
$$\bar{\mathcal{M}} = \{\mathbf{Q} \in \mathcal{Q} : \sum_{i=1}^n f_i(Q_i) < K\}.$$

Thus, $\mathcal{Q}$ is the set of possible cumulative production vectors, $\mathcal{M}$ is the subset of $\mathcal{Q}$ where the oil can be produced at the maximum rate K , and $\bar{\mathcal{M}}$ is the subset of $\mathcal{Q}$ where the oil cannot be produced at the maximum rate K .



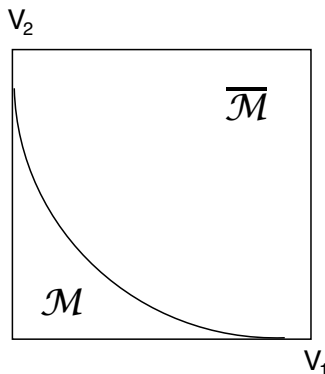
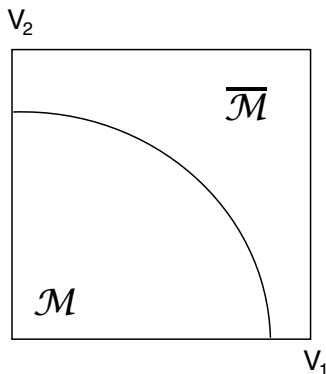
Optimizing production strategies

The sets $\mathcal{Q}$, $\mathcal{M}$ and $\bar{\mathcal{M}}$:



Optimizing production strategies

We shall see that the solution to the optimization problem depends on the shape of $\mathcal{M}$ and $\overline{\mathcal{M}}$:



Optimizing production strategies

Proposition

Consider a field with n reservoirs with PPR-functions $f_1, \dots, f_n$.

- (i) If $f_1, \dots, f_n$ are convex, the set $\bar{\mathcal{M}}$ is convex.*
- (ii) If $f_1, \dots, f_n$ are concave, the set $\mathcal{M}$ is convex.*

NOTE: If $\bar{\mathcal{M}}$ is convex, then $\bar{\mathcal{M}} \cup \partial(\mathcal{M})$ is convex as well. Similarly, if $\mathcal{M}$ is convex, then $\mathcal{M} \cup \partial(\mathcal{M})$ is convex as well.



Optimizing production strategies

Assume first that the PPR-functions are convex, and let $\mathbf{Q}^1 = (Q_1^1, \dots, Q_n^1)$ and $\mathbf{Q}^2 = (Q_1^2, \dots, Q_n^2)$ be two vectors in $\bar{\mathcal{M}}$. Thus, we have:

$$\sum_{i=1}^n f_i(Q_i^j) < K, \quad j = 1, 2.$$

Then let $0 \leq \alpha \leq 1$, and consider the vector $\mathbf{Q} = (Q_1, \dots, Q_n) = \alpha \mathbf{Q}^1 + (1 - \alpha) \mathbf{Q}^2$. Since the PPR-functions are convex, we have:

$$\begin{aligned} \sum_{i=1}^n f_i(Q_i) &= \sum_{i=1}^n f_i(\alpha Q_i^1 + (1 - \alpha) Q_i^2) \\ &\leq \alpha \sum_{i=1}^n f_i(Q_i^1) + (1 - \alpha) \sum_{i=1}^n f_i(Q_i^2) < K \end{aligned}$$

Thus, we conclude that $\mathbf{Q} \in \bar{\mathcal{M}}$ as well. Hence $\bar{\mathcal{M}}$ is convex.



Optimizing production strategies

Assume then that the PPR-functions are concave, and let $\mathbf{Q}^1 = (Q_1^1, \dots, Q_n^1)$ and $\mathbf{Q}^2 = (Q_1^2, \dots, Q_n^2)$ be two vectors in $\mathcal{M}$. Thus, we have:

$$\sum_{i=1}^n f_i(Q_i^j) \geq K, \quad j = 1, 2.$$

Then let $0 \leq \alpha \leq 1$, and consider the vector

$\mathbf{Q} = (Q_1, \dots, Q_n) = \alpha \mathbf{Q}^1 + (1 - \alpha) \mathbf{Q}^2$. Since the PPR-functions are concave, we have:

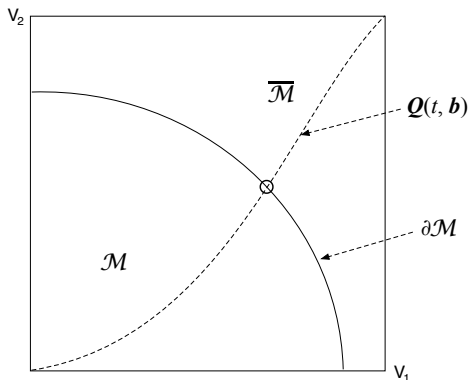
$$\begin{aligned} \sum_{i=1}^n f_i(Q_i) &= \sum_{i=1}^n f_i(\alpha Q_i^1 + (1 - \alpha) Q_i^2) \\ &\geq \alpha \sum_{i=1}^n f_i(Q_i^1) + (1 - \alpha) \sum_{i=1}^n f_i(Q_i^2) \geq K \end{aligned}$$

Thus, we conclude that $\mathbf{Q} \in \mathcal{M}$ as well. Hence $\mathcal{M}$ is convex.



Optimizing production strategies

Let $\mathbf{b}$ be any production strategy, and consider the points in $\mathcal{Q}$ generated by $\mathbf{Q}(t, \mathbf{b})$ as t increases.



From the boundary conditions we know that $\mathbf{Q}(0, \mathbf{b}) = \mathbf{0}$. Furthermore, $\mathbf{Q}(t, \mathbf{b})$ will move along some path in $\mathcal{M}$ until the boundary $\partial(\mathcal{M})$ is reached.



Optimizing production strategies

We denote the path $\{\mathbf{Q}(t, \mathbf{b}) : 0 \leq t < \infty\}$ by $\mathcal{P}(\mathbf{b})$.

If $\mathbf{b} \in \mathcal{B}$, $\mathcal{P}(\mathbf{b})$ is said to be a *valid path*, while if $\mathbf{b} \in \mathcal{B}'$, $\mathcal{P}(\mathbf{b})$ is called an *admissible path*.

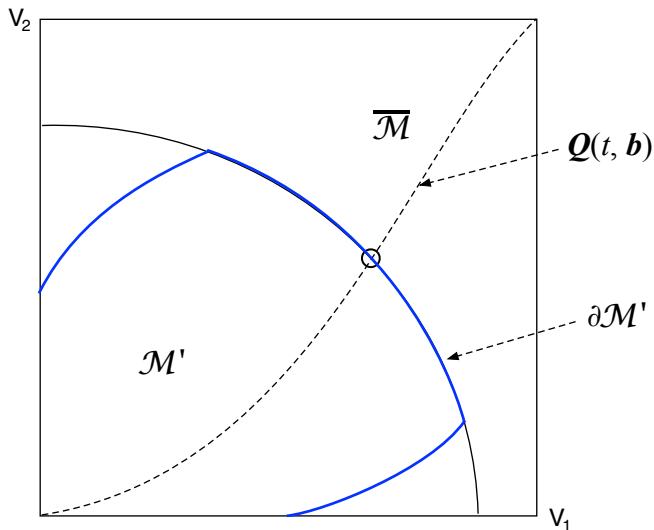
In general only a subset of $\mathcal{M}$ can be reached by admissible paths. We denote this subset by $\mathcal{M}'$.

Let $\partial(\mathcal{M}') = \partial(\mathcal{M}) \cap \mathcal{M}'$.

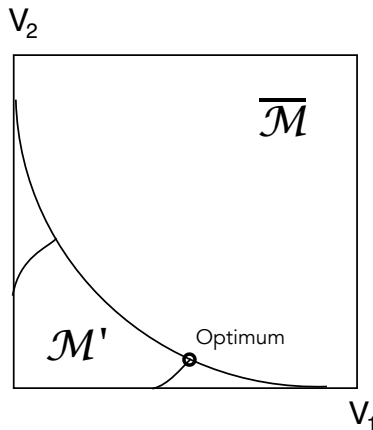
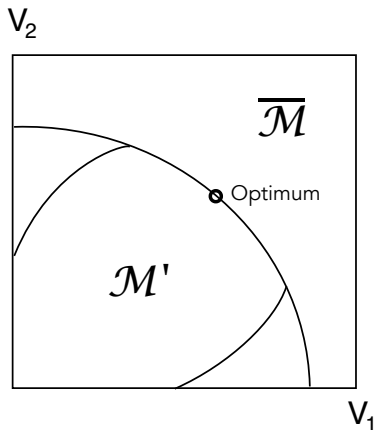
We assume that all points in $\partial(\mathcal{M}')$ are reachable by admissible paths.



Optimizing production strategies



Optimizing production strategies



Optimizing production strategies

Algorithm

Let ϕ be a monotone, symmetric objective function. Then a production strategy $\mathbf{b}$ which is optimal with respect to ϕ can be found as follows:

STEP 1. *Find $\mathbf{Q}^{opt} \in \partial(\mathcal{M}')$ such that $\phi(\mathbf{Q}^{opt}) \geq \phi(\mathbf{Q})$ for all $\mathbf{Q} \in \partial(\mathcal{M}')$.*

STEP 2. *Find a production strategy $\mathbf{b} \in \mathcal{B}'$ such that $\mathbf{Q}(T_K(\mathbf{b})) = \mathbf{Q}^{opt}$.*



Priority strategies

We now focus on the case where the PPR-functions are *convex*. Thus, the optimal strategy have an outcrossing point at the boundary of $\partial(\mathcal{M}')$. Such strategies are in a sense *extreme strategies* in the way the reservoirs are prioritized.

Definition

Consider a field with n reservoirs with PPR-functions $f_1, \dots, f_n$, and let $\pi = (\pi_1, \dots, \pi_n)$ be a permutation vector representing the prioritization order of the reservoirs.

The priority strategy relative to π , denoted $\mathbf{b}^\pi$, is defined by letting the production rates at time t , $q_1(t), \dots, q_n(t)$, be given by:

$$q_{\pi_i}(t) = \min[f_{\pi_i}(Q_{\pi_i}(t)), K - \sum_{j < i} q_{\pi_j}(t)], \quad i = 1, \dots, n.$$

Priority strategies

We observe that when assigning the production rate $q_{\pi_i}(t)$ to reservoir π_i , this is limited by $K - \sum_{j < i} q_{\pi_j}(t)$, i.e., the remaining processing capacity after assigning production rates to all the reservoirs with higher priority.

– If $f_{\pi_i}(Q_{\pi_i}(t)) \leq K - \sum_{j < i} q_{\pi_j}(t)$, reservoir π_i can be produced without any choking, and the remaining processing capacity is passed on to the reservoirs with lower priorities.

– If on the other hand $f_{\pi_i}(Q_{\pi_i}(t)) > K - \sum_{j < i} q_{\pi_j}(t)$, the production at reservoir π_i is choked so that $q_{\pi_i}(t) = K - \sum_{j < i} q_{\pi_j}(t)$. Thus, in this case *all the remaining processing capacity* is used on this reservoir, and nothing is passed on to the reservoirs with lower priorities.



Priority strategies

We introduce the following quantities ($i = 1, \dots, n$):

$$T_i = T_i(\mathbf{b}^\pi) = \inf\{t \geq 0 : \sum_{j=1}^i f_{\pi_j}(Q_{\pi_j}(t, \mathbf{b}^\pi)) < K\}.$$

We also let $T_0 = 0$, and note that we obviously have:

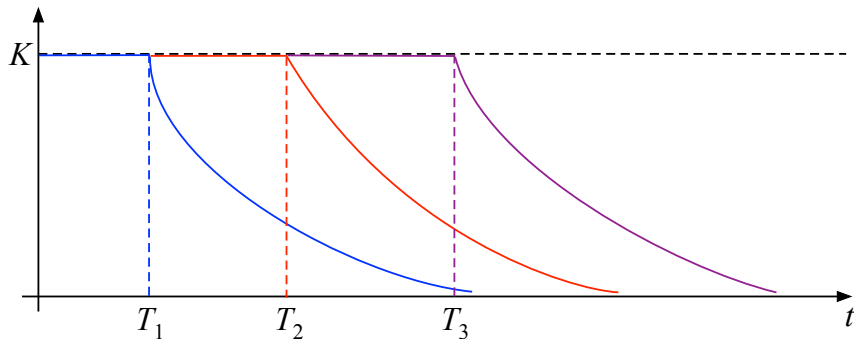
$$0 = T_0 \leq T_1 \leq \dots \leq T_n = T_K(\mathbf{b}^\pi).$$

Thus, $T_1, \dots, T_n$ defines an increasing sequence of *subplateau sets*, $[0, T_1], \dots, [0, T_n]$, where the last one is equal to the plateau interval $[0, T_K(\mathbf{b}^\pi)]$.

$T_1, \dots, T_n$ are called the *subplateau lengths* for the given priority strategy.



Priority strategies



Priority strategies

We now let $i \in \{1, \dots, n\}$, and assume that $T_{i-1} < t < T_i$. Then the reservoirs $\pi_1, \dots, \pi_{i-1}$ are produced without choking, i.e.:

$$q_{\pi_j}(t) = f_{\pi_j}(Q_{\pi_j}(t)), \quad j = 1, \dots, i-1.$$

Furthermore, the reservoir π_i is produced *with choking* so that:

$$q_{\pi_i}(t) = K - \sum_{j < i} q_{\pi_j}(t) = K - \sum_{j < i} f_{\pi_j}(Q_{\pi_j}(t)).$$

Finally the reservoirs $\pi_{i+1}, \dots, \pi_n$ are not produced at all.

NOTE: For $t \geq T_i$ we have:

$$f_{\pi_i}(Q_{\pi_i}(t)) \leq K - \sum_{j < i} q_{\pi_j}(t) = K - \sum_{j < i} f_{\pi_j}(Q_{\pi_j}(t)).$$

Thus, from this point of time the reservoir π_i can be produced without any choking. Thus, for $t \geq T_i$ we have $q_{\pi_i}(t) = f_{\pi_i}(Q_{\pi_i}(t))$.



Priority strategies

Summarizing this we see that for $i = 1, \dots, n$, the production rate, $q_{\pi_i}(t)$ is given by:

$$q_{\pi_i}(t) = \begin{cases} 0 & \text{if } t < T_{i-1}, \\ K - \sum_{j < i} f_{\pi_j}(Q_{\pi_j}(t)) & \text{if } T_{i-1} \leq t < T_i, \\ f_{\pi_i}(Q_{\pi_i}(t)) & \text{if } t \geq T_i. \end{cases}$$

We recall that if π is a permutation vector, the corresponding priority strategy is denoted by $\mathbf{b}^\pi$.

The class of all priority strategies is denoted by $\mathcal{B}^{PR}$.



Priority strategies

Theorem

Consider a field with n reservoirs with convex PPR-functions $f_1, \dots, f_n$. Furthermore, let ϕ be a symmetric, monotone objective function.

Then, under certain mild restrictions an optimal production strategy can be found within the class $\mathcal{B}^{PR}$.

NOTE: As there are exactly $n!$ possible permutation vectors π , this means that the class $\mathcal{B}^{PR}$ is finite. Thus, in order to find the optimal strategy, only a finite number of strategies need to be analysed.

If n is large, searching through all $n!$ strategies is very time-consuming. In such cases, however, the search can be done iteratively by starting out with a random permutation, and then at each step always choose the next permutation so that the value of objective function is increased.



Optimization with linear PPR-functions

Consider a field with n reservoirs with PPR-functions $f_1, \dots, f_n$, such that:

$$f_i(Q_i(t)) = D_i(V_i - Q_i(t)), \quad i = 1, \dots, n,$$

where $V_1, \dots, V_n$ denotes the recoverable volumes from the n reservoirs, and where we assume that the reservoirs have been indexed so that $0 < D_1 \leq D_2 \leq \dots \leq D_n$.

NOTE: For a given linear PPR function $f_i(Q_i(t)) = D_i(V_i - Q_i(t))$, the maximum production rate at any given point of time is a **fixed fraction** D_i of the **remaining volume**, $(V_i - Q_i(t))$.

The factor D_i is referred to as the **decline factor** of the i th reservoir, $i = 1, \dots, n$.



Optimization with linear PPR-functions

Consider the i th reservoir, and let $T \geq 0$. If this reservoir is produced without any choking, i.e., with a choking factor function $b_i(t) = 1$ for all $t \geq T$, we can easily solve the **linear differential equation**:

$$q_i(t) = D_i(V_i - Q_i(t)), \quad t \geq T.$$

with the boundary condition $Q_i(t) = Q_i(T)$, and the solution is given by:

$$q_i(t) = D_i(V_i - Q_i(T)) \exp(-D_i(t - T)), \quad t \geq T.$$

Moreover, by integrating $q_i(t)$ from T to t we also get:

$$Q_i(t) = V_i(1 - e^{-D_i(t-T)}) + Q_i(T)e^{-D_i(t-T)}, \quad t \geq T.$$

NOTE: $Q_i(t)$ is expressed as a **convex combination** of V_i and $Q_i(T)$. As t **increases** the weight associated with V_i **increases** and the weight associated with $Q_i(T)$ **decreases**. Thus, as t **increases**, the cumulative volume approaches the **ultimate recoverable volume** V_i .



A result on dominating sums

Lemma

Assume that $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ are such that:

$$\sum_{i=1}^k x_i \geq \sum_{i=1}^k y_i, \quad k = 1, \dots, n.$$

Then for any $\mathbf{a} \in \mathbb{R}^n$ such that:

$$a_1 \geq a_2 \geq \dots \geq a_n \geq 0,$$

we also have:

$$\sum_{i=1}^k x_i a_i \geq \sum_{i=1}^k y_i a_i, \quad k = 1, \dots, n.$$



A result on dominating sums (cont.)

PROOF: Induction on k .

By the assumption it follows that $x_1 \geq y_1$, and since all the a_i s are assumed to be nonnegative:

$$x_1 a_1 \geq y_1 a_1.$$

Hence, the result obviously holds for $k = 1$.

We then assume that the result is proved for $k \leq m$, and consider the case where $k = m + 1$. We then introduce $b_i = a_i - a_{m+1}$, $i = 1, \dots, m$.

It follows that:

$$b_1 \geq b_2 \geq \dots \geq b_m \geq 0.$$

Hence, by the induction hypothesis we have that:

$$\sum_{i=1}^m x_i b_i \geq \sum_{i=1}^m y_i b_i.$$



A result on dominating sums (cont.)

By combining this we get that:

$$\begin{aligned}\sum_{i=1}^{m+1} x_i a_i &= \sum_{i=1}^m x_i (a_i - a_{m+1}) + a_{m+1} \sum_{i=1}^{m+1} x_i \\&= \sum_{i=1}^m x_i b_i + a_{m+1} \sum_{i=1}^{m+1} x_i \\&\geq \sum_{i=1}^m y_i b_i + a_{m+1} \sum_{i=1}^{m+1} y_i \\&= \sum_{i=1}^m y_i (a_i - a_{m+1}) + a_{m+1} \sum_{i=1}^{m+1} y_i \\&= \sum_{i=1}^{m+1} y_i a_i.\end{aligned}$$

Thus, the result holds for $k = m + 1$ as well, and hence for $k = 1, \dots, n$ by induction.



Optimization with linear PPR-functions

Theorem

Consider a field with n reservoirs with linear PPR-functions $f_1, \dots, f_n$ with decline factors $0 < D_1 \leq D_2 \leq \dots \leq D_n$.

Then let $\mathbf{b}^1$ denote the priority strategy corresponding to the permutation $\pi = (1, 2, \dots, n)$, and let $\mathbf{b}^2$ be any other valid production strategy.

Then $Q(t, \mathbf{b}^1) \geq Q(t, \mathbf{b}^2)$ for all $t \geq 0$.

Thus, $\mathbf{b}^1$ is optimal with respect to any monotone, symmetric objective function.



Optimization with linear PPR-functions

PROOF: We introduce the plateau lengths $T_1, \dots, T_n$.

If the priority strategy $\mathbf{b}^1$ is used, we get the following:

Reservoir 1 is produced at the rate K throughout the interval $[0, T_1]$ and will be produced without any choking for $t \geq T_1$.

Reservoirs 1 and 2 are produced at a total rate K throughout the interval $[0, T_2]$ and will be produced without any choking for $t \geq T_2$.

...



Optimization with linear PPR-functions

We shall now prove by induction that:

$$\sum_{j=1}^i Q_j(t, \mathbf{b}^1) \geq \sum_{j=1}^i Q_j(t, \mathbf{b}^2), \quad t \geq 0, \quad i = 1, \dots, n.$$

Thus, we start out by considering the case where $i = 1$, and assume that the priority strategy $\mathbf{b}^1$ is used.

If $0 \leq t \leq T_1$, then obviously:

$$Q_1(t, \mathbf{b}^1) = Kt.$$

If $t > T_1$, we know that reservoir 1 is produced without any choking. Thus, we have:

$$Q_1(t, \mathbf{b}^1) = V_1(1 - e^{-D_1(t-T_1)}) + Q_1(T_1, \mathbf{b}^1)e^{-D_1(t-T_1)}.$$



Optimization with linear PPR-functions

We then consider the situation where $\mathbf{b}^2$ is used instead.

If $0 \leq t \leq T_1$, then obviously:

$$Q_1(t, \mathbf{b}^2) \leq Kt = Q_1(t, \mathbf{b}^1).$$

If $t > T_1$, we have:

$$Q_1(t, \mathbf{b}^2) \leq V_1(1 - e^{-D_1(t-T_1)}) + Q_1(T_1, \mathbf{b}^2)e^{-D_1(t-T_1)}.$$

Thus, since $Q_1(T_1, \mathbf{b}^1) \geq Q_1(T_1, \mathbf{b}^2)$, it follows that:

$$Q_1(t, \mathbf{b}^1) \geq Q_1(t, \mathbf{b}^2) \text{ for all } t > T_1.$$

Hence, we conclude that $Q_1(t, \mathbf{b}^1) \geq Q_1(t, \mathbf{b}^2)$ for all $t \geq 0$, i.e., the induction hypothesis is proved for $i = 1$.



Optimization with linear PPR-functions

We then assume that the induction hypothesis is proved for $i = 1, \dots, (k - 1)$, and consider the case where $i = k$.

If $0 \leq t \leq T_k$, we have:

$$\sum_{j=1}^k Q_j(t, \mathbf{b}^1) = Kt \geq \sum_{j=1}^k Q_j(t, \mathbf{b}^2).$$

In particular:

$$\sum_{j=1}^k Q_j(T_k, \mathbf{b}^1) \geq \sum_{j=1}^k Q_j(T_k, \mathbf{b}^2). \quad (*)$$



Optimization with linear PPR-functions

We then consider the case where $t > T_k$.

If $\mathbf{b}^1$ is used, the reservoirs $1, 2, \dots, k$ are produced without any choking, thus:

$$\begin{aligned}\sum_{j=1}^k Q_j(t, \mathbf{b}^1) &= \sum_{j=1}^k V_j(1 - e^{-D_j(t-T_k)}) \\ &\quad + \sum_{j=1}^k Q_j(T_k, \mathbf{b}^1)e^{-D_j(t-T_k)}.\end{aligned}$$



Optimization with linear PPR-functions

If, on the other hand, $\mathbf{b}^2$ is used, we get:

$$\begin{aligned} \sum_{j=1}^k Q_j(t, \mathbf{b}^2) &\leq \sum_{j=1}^k V_j(1 - e^{-D_j(t-T_k)}) \\ &\quad + \sum_{j=1}^k Q_j(T_k, \mathbf{b}^2)e^{-D_j(t-T_k)}. \end{aligned}$$



Optimization with linear PPR-functions

By the induction hypothesis and (*) we have that:

$$\sum_{j=1}^i Q_j(T_k, \mathbf{b}^1) \geq \sum_{j=1}^i Q_j(T_k, \mathbf{b}^2), \quad i = 1, \dots, k.$$

Moreover, since $D_1 \leq D_2 \leq \dots \leq D_k$, we have:

$$e^{-D_1(t-T_k)} \geq \dots \geq e^{-D_k(t-T_k)}, \text{ for all } t \geq T_k.$$

Then it follows by the lemma on dominating sums that:

$$\sum_{j=1}^k Q_j(T_k, \mathbf{b}^1) e^{-D_j(t-T_k)} \geq \sum_{j=1}^k Q_j(T_k, \mathbf{b}^2) e^{-D_j(t-T_k)}$$



Optimization with linear PPR-functions

Hence, also for $t \geq T_k$ we have:

$$\begin{aligned}\sum_{j=1}^k Q_j(t, \mathbf{b}^2) &\leq \sum_{j=1}^k V_j(1 - e^{-D_j(t-T_k)}) \\ &\quad + \sum_{j=1}^k Q_j(T_k, \mathbf{b}^2)e^{-D_j(t-T_k)} \\ &\leq \sum_{j=1}^k V_j(1 - e^{-D_j(t-T_k)}) \\ &\quad + \sum_{j=1}^k Q_j(T_k, \mathbf{b}^1)e^{-D_j(t-T_k)} \\ &= \sum_{j=1}^k Q_j(t, \mathbf{b}^1).\end{aligned}$$



Optimization with linear PPR-functions

By combining all this, we get for all $t \geq 0$ that:

$$\sum_{j=1}^k Q_j(t, \mathbf{b}^1) \geq \sum_{j=1}^k Q_j(t, \mathbf{b}^2).$$

Thus, the induction hypothesis is proved for $i = k$ as well.

Hence, the result is proved by induction

