UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: STK4400 — Risk and reliability analysis

Day of examination: May 3, 2023

Examination hours: –

This problem set consists of 4 pages.

Appendices: None

Permitted aids: All lecture notes.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

A cedent has m non-negative *absolutely continuously* distributed risks:

$$X_1, \ldots, X_m$$

with densities $f_{X_1}(x), \ldots, f_{X_m}(x)$ respectively. We also introduce:

$$S_{X_i}(x) = P(X_i > x), \quad i = 1, \ldots, m,$$

and assume that $S_{X_i}(x)$ is strictly decreasing for all $x \geq 0$, for $i = 1, \ldots, m$.

All risks are reinsured using *insurance layer* contracts:

$$R_i(X_i) = \begin{cases} 0 & X_i < a_i \\ X_i - a_i & a_i \leq X_i \leq b_i \\ b_i - a_i & X_i > b_i \end{cases} \qquad i = 1, \ldots, m.$$

where $0 < a_i < b_i$ are the *contract parameters* for the ith risk, $i = 1, \ldots, m$. As a result, the *retained risks* covered by the cedent are:

$$I_i(X_i) = \begin{cases} X_i & X_i < a_i \\ a_i & a_i \leq X_i \leq b_i \\ X_i - (b_i - a_i) & X_i > b_i \end{cases} \qquad i = 1, \ldots, m.$$

The price paid by the cedent for the ith contract is assumed to be of the form:

$$(1 + \theta)\, \mathrm{E}[R_i(X_i)], \quad i = 1, \ldots, m.$$

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where $\theta > 0$ represents the risk premiums charged by the reinsurance company for handling the risks. The total risk covered by the cedent is then:

$$\Psi = \sum_{i=1}^{m} I_i(X_i) + (1+\theta) \sum_{i=1}^{m} \mathrm{E}[R_i(X_i)].$$

(a) Explain briefly why that the resulting total α-level value-at-risk, i.e., the upper α-percentile of the distribution of Ψ, is given by:

$$\mathrm{V@R}_{\alpha}(\Psi) = \mathrm{V@R}_{\alpha}(\sum_{i=1}^{m} I_i(X_i)) + (1+\theta) \sum_{i=1}^{m} \mathrm{E}[R_i(X_i)].$$

(b) We want to find contract parameters, $a_1, \ldots, a_m$ and $b_1, \ldots, b_m$ so that the objective function, $\mathrm{V@R}_{\alpha}(\Psi)$, is minimized. Optimal values for $a_1, \ldots, a_m$, are given by

$$a_i = S_{X_i}^{-1}(\frac{1}{1+\theta}), \quad i = 1, \ldots, m.$$

Explain briefly why the corresponding optimal values $b_1, \ldots, b_m$ can be found by solving the following optimization problem:

$$\begin{aligned} \text{Minimize:} \quad & \Phi = \sum_{i=1}^{m} \Phi_i = \sum_{i=1}^{m} \mathrm{E}[R_i(X_i)] \\ \text{Subject to:} \quad & P(\sum_{i=1}^{m} I_i(X_i) > \sum_{i=1}^{m} a_i) = \alpha. \end{aligned}$$

with respect to $b_1, \ldots, b_m$, where we have introduced $\Phi_i = \mathrm{E}[R_i(X_i)], \quad i = 1, \ldots, m$.

(c) Instead of finding the optimal contract parameters $b_1, \ldots, b_m$ directly, it is easier to solve the optimization problem with respect to $B_1, \ldots, B_m$, where $B_i = S_{X_i}(b_i)$, $i = 1, \ldots, m$. It can be shown that:

$$\frac{\partial \Phi_i}{\partial B_i} = -\frac{B_i}{f_{X_i}(S_{X_i}^{-1}(B_i))}, \quad i = 1, \ldots, m.$$

Show that Φ_i is a *convex function* of B_i if X_i has a *decreasing hazard rate*, and that Φ_i is a *concave function* of B_i if X_i has a *increasing hazard rate*.

(d) In this point we consider two examples where $m = 2$. In both examples $\alpha = 0.01$ while $\theta = 0.20$.

In the first example X_1 and X_2 are independent and Pareto-distributed with $\mathrm{E}[X_i] = 50$ and $\mathrm{SD}[X_i] = 70$, $i = 1, 2$. Figure 1 shows iso-curves for the function $\Phi = \Phi(B_1, B_2)$ as well as the constraint curve for this example.

In the second example X_1 and X_2 are independent and truncated normally distributed with $\mathrm{E}[X_i] = 50$ and $\mathrm{SD}[X_i] = 30$, $i = 1, 2$. Figure 2 shows iso-curves for the function $\Phi = \Phi(B_1, B_2)$ as well as the constraint curve for this example.

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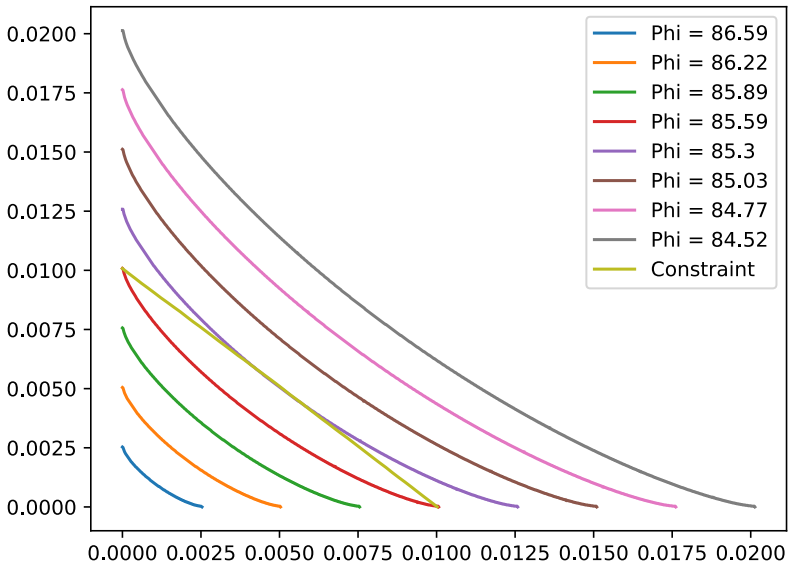


Figure 1: Iso-curves and constraint curve for two Pareto distributions where $\text{Corr}(X_1, X_2) = 0.0$.

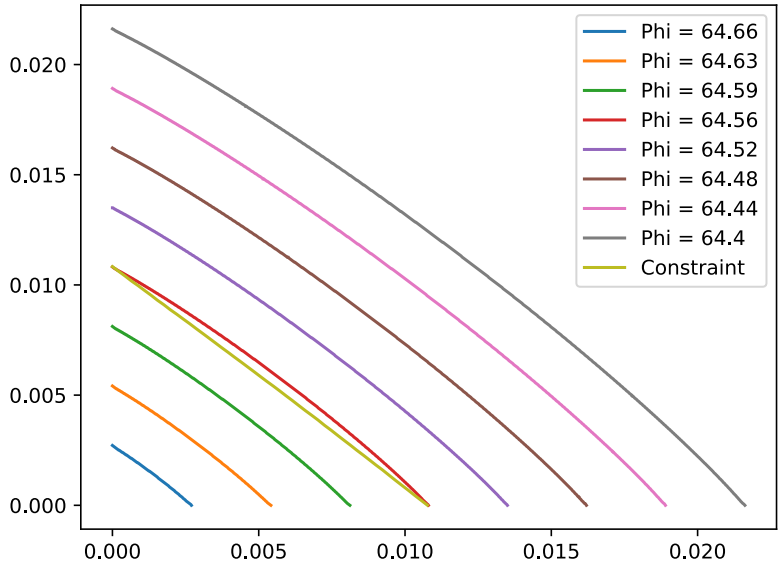


Figure 2: Iso-curves and constraint curve for two truncated normal-distributions where $\text{Corr}(X_1, X_2) = 0.0$.

Discuss the shape of the iso-curves in the two examples in the light of (c). Identify the optimal solutions on the two plots. No detailed calculations are required.

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(e) In this point we consider the second example from (d). However, in this case X_1 and X_2 are correlated with $\text{Corr}(X_1, X_2) = 0.8$. Figure 3 shows iso-curves for the function $\Phi = \Phi(B_1, B_2)$ as well as the constraint curve for this example. By considering the plot discuss how the correlation affects the optimal solution.

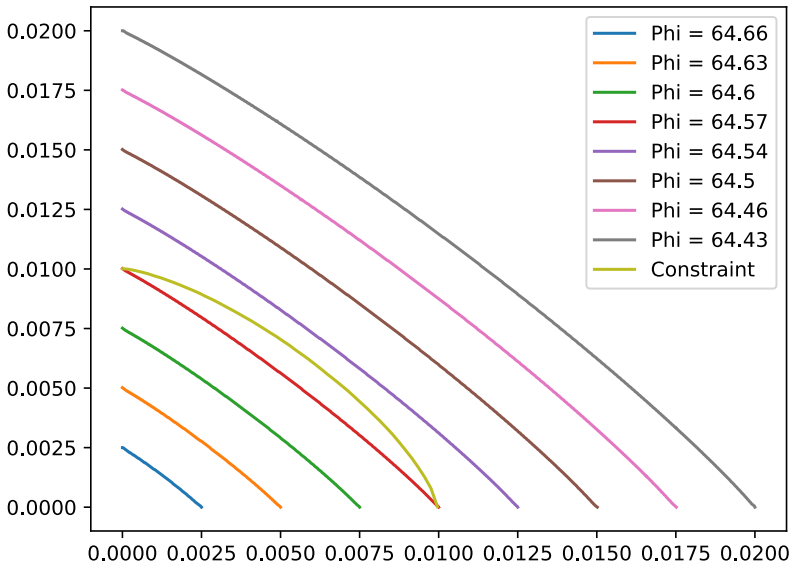


Figure 3: Iso-curves and constraint curve for two truncated normal-distributions where $\text{Corr}(X_1, X_2) = 0.8$.

END