UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Examination in STK4500 — Life Insurance and Finance.

Day of examination: Tuesday, June 10, 2014.

Examination hours: 14.30 – 18.30.

This problem set consists of 5 pages.

Appendices: Formulary

Permitted aids: Approved calculator.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1. (10 points)

Consider a permanent disability model. In this model the state of the insured $X_t \in S$ is modeled by a regular Markov chain with state space $S = \{*, \diamondsuit, \dagger\}$, where $*$ ="active", $\diamondsuit$ ="disabled" and $\dagger$ ="dead".
Suppose that the transition rates for this model are given by the following constants:

$$\mu_{*\diamondsuit}(t) = 0.0013, \mu_{*\dagger}(t) = 0.003, \mu_{\diamondsuit\dagger}(t) = \mu_{*\dagger}(t).$$

(i) Find explicit formulas for the transition probabilities $p_{ij}(s,t), i, j \in S$.

(ii) Calculate $p_{**}(x, x+20)$ and $p_{*\diamondsuit}(x, x+20)$ for $x = 30$ (years).

Problem 2. (10 points)

Consider a permanent disability insurance (in discrete time). Let $x = 30$ years be the initial age of the insured and let 50 be the age at maturity. Further assume that $\delta = 3.5\%$ (interest rate intensity) and that the yearly disability pension is given by 15000$. Suppose that the transition rates are given as in Problem 1.

Compute the prospective reserve of the disability pension payments at time $t = 47$ years, given that $X_t = *$.

Problem 3. (10 points)

An insurance company issues a 10−year unit-linked term insurance with a single premium of

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$P = 15000\$$ to a life aged 50. There is a deduction for initial expenses given by 3.5% and the rest of the premium is invested in an equity fund whose dynamics S_t of its values over time is described by the Black-Scholes model with $S_0 = 1$. Further, management charges are deducted on a daily basis from the insured's account at a rate of $\beta = 0.5\%$ per year (i.e. in the sense of a continuous deduction based on the discount factor $e^{-\beta t}$). If death occurs during the contract period a death benefit of 115% of the fund value is provided.
Assume that
(i) the transition rate is constant and given by

$$\mu_{*\dagger}(t) = 0.009$$

(ii) the risk free rate of interest is $r = 4\%$ per year, continuously compounded.
(iii) the volatility of S_t is $\sigma = 22\%$ per year.

Compute the prospective reserve of the benefits at time $t = 0$.

Problem 4. (10 points)

Consider a 10-year pure endowment issued to a life aged 50. The endowment amount, which is paid in the case of survival, is given by 100000\$. Assume for this policy stochastic interest rates $r(t)$ described by the Vasicek model with parameters $r(0) = 0.03, a = 0.5, b = 0.03$ and $\sigma = 0.012$. Let $\lambda = -1$ (risk premium) and

$$\mu_{*\dagger}(t) = 0.009$$

be the constant transition rate.

(i) Calculate the prospective reserve of the endowment payment at time $t = 0$.

(ii) Explain how to find the constant continuously paid yearly premiums P of this policy based on the equivalence principle.

Problem 5. (5 points)

Consider a contingent claim with payoff

$$X := \max(0, \frac{1}{T}\int_0^T S(t)dt - K),$$

at maturity T, where $S(t)$ is the stock price process described by the Black-Scholes model and $K > 0$ is the strike.
(i) Show that the process

$$Y(t) := \frac{1}{S(t)}(\frac{1}{T}\int_0^t S(u)du - K)$$

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satisfies the stochastic equation

$$dY(t) = (\frac{1}{T} + (\sigma^2 - r)Y(t))dt - \sigma Y(t)d\widetilde{B}_t,$$

where $\widetilde{B}_t$ is a Brownian motion with respect to the equivalent martingale measure $\widetilde{P}$.

(ii) Show that the replicating portfolio $V(t)$ of X can be written as

$$V(t) = \exp(-r(T-t))S(t)F(t, Y(t))$$

for a function $F(t, y)$.

End

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Appendix: Formulary

a) Forward Kolmogorov equation:

$$\frac{d}{dt}p_{ij}(s,t) = -p_{ij}(s,t)\mu_j(t) + \sum_{k\neq j} p_{ik}(s,t)\mu_{kj}(t), i,j \in S$$

with $p_{ij}(s,s) = 0$, if $i \neq j$, $p_{ij}(s,s) = 1$, if $i = j$, where $\mu_j(t) = \sum_{k\neq j} \mu_{jk}(t)$.

b)

$$p_{jj}(s,t) = \overline{p}_{jj}(s,t) = \exp(-\sum_{k\neq j}\int_s^t \mu_{jk}(u)du), j \in S.$$

c) Thiele's difference equation:

$$V_i^+(t) = a_i^{\mathrm{Pre}}(t) + \sum_{j\in S} e^{-\delta}\cdot p_{ij}(t,t+1)\cdot\{a_{ij}^{\mathrm{Post}}(t) + V_j^+(t+1)\},$$

where $a_i^{\mathrm{Pre}}(t)$ (pension payments) and $a_{ij}^{\mathrm{Post}}(t)$ (benefit payments) are the policy functions.

d) Prospective reserve (in continuous time):

$$\begin{aligned} V_j^+(t) &= \frac{1}{v(t)}\{\sum_{g\in S}\int_{(t,\infty)} v(s)p_{jg}(t,s)da_g(s) \\ &\quad + \sum_{g\in S}\int_{(t,\infty)} v(s)p_{jg}(t,s)(\sum_{\substack{h\in S,\\ h\neq g}} a_{gh}(s)\cdot\mu_{gh}(s))ds\}, \end{aligned}$$

for $j \in S$.

e) Black-Scholes model:

$$S_t = x + \int_0^t \mu S_u du + \int_0^t \sigma S_u dB_u, 0 \leq t \leq T,$$

where $\sigma \neq 0$ and μ are constants and where $B_t, 0 \leq t \leq T$ is a Brownian motion.

f) pricing formula for a claim X:

$$\begin{aligned} ClaimValue_t &= E_{\widetilde{P}}[e^{-(T-t)\cdot r}X\,|\mathcal{G}_t]\,, 0 \leq t \leq T \\ ClaimValue_0 &= E_{\widetilde{P}}[e^{-(T-t)\cdot r}X], \end{aligned}$$

where $\widetilde{P}$ (equivalent martingale measure) is the probability measure such that $\widetilde{S}_t = e^{-rt}S_t, 0 \leq t \leq T$ (in the Black-Scholes model) is a martingale under $\widetilde{P}$.

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g) Vasicek model:

$$r(t) = x + \int_0^t a(b - r(u))du + \sigma B_t$$

for non-negative constants a, b and σ.

h) Bond value at time $t = 0$ (in the Vasicek model):

$$P(0,T) = \exp(-T \cdot R(T, r(0))),$$

where

$$\begin{aligned} & R(s,x) \\ = \quad & (b - (\lambda\sigma)/a - \frac{\sigma^2}{2a^2}) - \frac{1}{a \cdot s}[((b - (\lambda\sigma)/a - \frac{\sigma^2}{2a^2}) - x)(1 - e^{-as}) - \frac{\sigma^2}{4a^2}(1 - e^{-as})^2]. \end{aligned}$$

i) Integration by parts formula:

$$X_t = X_0 + \int_0^t K_u du + \int_0^t H_u dB_u, Y_t = Y_0 + \int_0^t \widetilde{K}_u du + \int_0^t \widetilde{H}_u dB_u.$$

Then

$$X_t Y_t = X_0 Y_0 + \int_0^t X_u dY_u + \int_0^t Y_u dX_u + [X,Y]_t,$$

where

$$[X,Y]_t = \int_0^t H_u \widetilde{H}_u du.$$

j) Itô's formula:

$$f(X_t) = f(X_0) + \int_0^t \frac{d}{dx} f(X_u) dX_u + \frac{1}{2}\int_0^t \frac{d^2}{dx^2} f(X_u) \cdot (H_u)^2 du$$

for $X_t = X_0 + \int_0^t K_u du + \int_0^t H_u dB_u$.