UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Examination in STK4500 — Life Insurance and Finance.

Day of examination: Monday, June 08, 2015.

Examination hours: 14.30 – 18.30.

This problem set consists of 5 pages.

Appendices: Formulary

Permitted aids: Approved calculator.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1. (10 points)

Let us have a look at a permanent disability model. In this model the state of the insured $X_t \in S$ is modeled by a regular Markov chain with state space $S = \{*, \diamondsuit, \dagger\}$, where $*$ ="active", $\diamondsuit$ ="disabled" and $\dagger$ ="dead".
Suppose that the transition rates for this model are given by the following constants:

$$\mu_{*\diamondsuit}(t) = 0.002, \mu_{*\dagger}(t) = 0.0045, \mu_{\diamondsuit\dagger}(t) = \mu_{*\dagger}(t).$$

(i) Find explicit formulas for the transition probabilities $p_{ij}(s,t), i, j \in S$.

(ii) Compute $p_{**}(x, x+20)$ and $p_{*\diamondsuit}(x, x+20)$ for $x = 40$ (years).

Problem 2. (10 points)

Consider a 10−year unit-linked term insurance with a single premium of $P = 12500\$$ issued to a life aged 55. There is an initial deduction for expenses given by 3% and the rest of the premium is invested in an equity fund whose dynamics S_t of its values over time is described by the Black-Scholes model with $S_0 = 1$. In addition, management charges are deducted on a daily basis from the account of the insured at a rate of $\beta = 0.4\%$ per year (i.e. in the sense of a continuous deduction based on the discount factor $e^{-\beta t}$). If death occurs during the contract period a death benefit of 125% of the fund value is paid.
Suppose that
(i) the transition rate is constant and given by

$$\mu_{*\dagger}(t) = 0.011$$

(ii) the risk free rate of interest is $r = 3\%$ per year, continuously compounded.

(iii) the volatility of S_t is $\sigma = 25\%$ per year.

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Calculate the prospective reserve of the benefits at time $t = 0$.

Problem 3. (10 points)

Let us consider a 10-year pure endowment issued to a life aged 45. The endowment amount, which is provided in the case of survival, is given by 130000\$. Suppose for this policy stochastic interest rates $r(t)$ described by the Vasicek model with parameters $r(0) = 0.025, a = 0.4, b = 0.035$ and $\sigma = 0.01$. Let $\lambda = -1$ (risk premium) and

$$\mu_{*\dagger}(t) = 0.004$$

be the constant transition rate.

Compute the prospective reserve of the endowment payment at time $t = 0$.

Problem 4. (10 points)

Consider a permanent disability insurance with term two years issued to a healthy life aged 55. Premiums are paid continuously during the contract period while the policyholder is in the healthy state. Disability benefits are paid at the rate of 50000\$ per year.
Suppose that the following transition probabilities are used for the insured:

$$\begin{aligned} p_{**}(55, 55+t) &= \frac{2}{3}\exp(-0.015t) + \frac{1}{3}\exp(-0.01t), \\ p_{*\dagger}(55, 55+t) &= 1 - \exp(-0.01t). \end{aligned}$$

(i) Let $\delta = 3.5\%$ be the interest rate intensity and calculate the annual premiums P by means of the equivalence principle.

(ii) Let $B_t, 0 \le t \le T$ be a 1−dimensional Brownian motion. Calculate the probability density of

$$X \stackrel{def}{=} \int_0^T B_s ds.$$

Problem 5. (6 points)

Consider a (1-dimensional) Black-Scholes market with constant interest rate r and a stock whose stock price $S_t^{(1)}$ at time t is modeled by

$$S_t^{(1)} = x + \int_0^t \mu S_u^{(1)} du + \int_0^t \sigma_1 S_u^{(1)} dB_u, 0 \le t \le T,$$

where $B_t, 0 \le t \le T$ is a Brownian motion and $\mu \in \mathbb{R}, \sigma_1 > 0$ are constants. Further, denote by $S_t^{(2)}$ be the stock price described by

$$S_t^{(2)} = x + \int_0^t \mu S_u^{(2)} du + \int_0^t \sigma_2 S_u^{(2)} dB_u, 0 \le t \le T$$

with respect to another Black-Scholes market with constant interest rate r, where $\sigma_2 > \sigma_1$ is a constant. Let p_i be the fair price at time $t = 0$ of a European call option with maturity T and strike price K on one unit $S_T^{(i)}$ for $i = 1, 2$.

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We now suppose a stock price process S_t of a Black-Scholes market with constant interest rate r satisfying

$$S_t = x + \int_0^t \mu S_u du + \int_0^t \widetilde{\sigma}(u) S_u dB_u, 0 \leq t \leq T,$$

where the volatility is *stochastic* and modeled by an adapted process $\widetilde{\sigma}(t), 0 \leq t \leq T$ with $\sigma_1 \leq \widetilde{\sigma}(t) \leq \sigma_2$ for all t.

The fair price p at time $t = 0$ of a European call option with maturity T and strike price K of S_T can be defined as

$$p = E_{P^*}[e^{-rT} \max(0, S_T - K)],$$

where P^* is the probability measure such that the discounted stock price becomes a martingale.

Show that

$$p \in [p_1, p_2].$$

Hint: Black-Scholes partial differential equation:

$$\begin{aligned} \frac{\partial C}{\partial t}(t,x) + \frac{\sigma^2 x^2}{2} \frac{\partial^2 C}{\partial x^2}(t,x) + rx \frac{\partial C}{\partial x}(t,x) - rC(t,x) &= 0, t \in [0,T), x > 0 \\ C(T,x) &= f(x). \end{aligned}$$

End

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Appendix: Formulary

a) Forward Kolmogorov equation:

$$\frac{d}{dt}p_{ij}(s,t) = -p_{ij}(s,t)\mu_j(t) + \sum_{k\neq j} p_{ik}(s,t)\mu_{kj}(t), i,j \in S$$

with $p_{ij}(s,s) = 0$, if $i \neq j$, $p_{ij}(s,s) = 1$, if $i = j$, where $\mu_j(t) = \sum_{k\neq j}\mu_{jk}(t)$.

b)

$$p_{jj}(s,t) = \overline{p}_{jj}(s,t) = \exp(-\sum_{k\neq j}\int_s^t \mu_{jk}(u)du), j \in S.$$

c) Prospective reserve (in continuous time):

$$\begin{aligned} V_j^+(t) &= \frac{1}{v(t)}\{\sum_{g\in S}\int_{(t,\infty)} v(s)p_{jg}(t,s)da_g(s) \\ &\quad + \sum_{g\in S}\int_{(t,\infty)} v(s)p_{jg}(t,s)(\sum_{\substack{h\in S,\\ h\neq g}} a_{gh}(s)\cdot\mu_{gh}(s))ds\}, \end{aligned}$$

for $j \in S$, where $a_i(t)$ (pension payments) and $a_{ij}(t)$ (benefit payments) are the policy functions.

d) Black-Scholes model:

$$S_t = x + \int_0^t \mu S_u du + \int_0^t \sigma S_u dB_u, 0 \leq t \leq T,$$

where $\sigma \neq 0$ and μ are constants and where $B_t, 0 \leq t \leq T$ is a Brownian motion.

e) pricing formula for a claim X:

$$\begin{aligned} ClaimValue_t &= E_{\widetilde{P}}[e^{-(T-t)\cdot r}X\,|\mathcal{G}_t]\,, 0 \leq t \leq T \\ ClaimValue_0 &= E_{\widetilde{P}}[e^{-(T-t)\cdot r}X], \end{aligned}$$

where $\widetilde{P}$ (equivalent martingale measure) is the probability measure such that $\widetilde{S}_t = e^{-rt}S_t, 0 \leq t \leq T$ (in the Black-Scholes model) is a martingale under $\widetilde{P}$.

f) Vasicek model:

$$r(t) = x + \int_0^t a(b - r(u))du + \sigma B_t$$

for non-negative constants a, b and σ.

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g) Bond value at time $t = 0$ (in the Vasicek model):

$$P(0,T) = \exp(-TR(T, r(0))),$$

where

$$\begin{aligned} & R(s,x) \\ = \quad & (b - (\lambda\sigma)/a - \frac{\sigma^2}{2a^2}) - \frac{1}{a \cdot s}[((b - (\lambda\sigma)/a - \frac{\sigma^2}{2a^2}) - x)(1 - e^{-as}) - \frac{\sigma^2}{4a^2}(1 - e^{-as})^2]. \end{aligned}$$

h) Probability density f of X:

$$P(X \leq y) = \int_{-\infty}^{y} f(z)dz$$

for all y.

i) European call option:

$$\max(0, S_T - K)$$