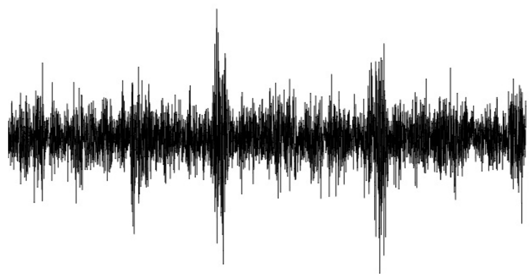
Exercises for STK 4500



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1 Diversification

We consider a portfolio of identical annuities on N lives, all of the same age x. The annuities are paid lifelong, annually, and in advance, and with a delay period of k years. The annual payment is constant and equal to 1. The cash value of the annuity of person i in the portfolio is given by the random variable

$$K_i = \sum_{t=k}^{\lfloor T_x^{(i)} \rfloor} v_t \cdot I(T_x^{(i)} \geq k)$$

where $T_x^{(i)}$ is the remaining life span of the person in question, the notation $\lfloor \xi \rfloor$ refers to the nearest integer smaller than or equal to ξ, and v_t is the discounted value today of a unit payable t years from now.

By Monte Carlo simulation you shall find an approximate probability distribution for the average cash value of the annuity payments in the portfolio, i.e. an approximate probability distribution for the stochastic variable

$$P = \frac{1}{N} \sum_{i=1}^{N} K_i$$

This should be carried out under two different ways of modelling the achieved return on the investments:

- Deterministically and with

$$\frac{v_{t+1}}{v_t} = \exp\left(-\mu\right), t = 1, 2, ..$$

- Stochastically and with

$$\frac{v_{t+1}}{v_t} = \exp\left(-\left(\mu - \frac{\sigma^2}{2}\right) - \sigma \cdot Z_t\right)\ , Z_1, Z_2, ...u.i.d. \sim N(0,1)$$

  Show that the expected reurn in the stochastic model equals the return in the deterministic model.

For the distribution of the life spans we assume a pure Gompertz death intensity such that the remaining life span can be simulated by means of the function

$$T_x = \frac{\log\left(1 - \frac{\log(c)\cdot\log(u)}{\beta\cdot c^x}\right)}{\log(c)}\ , u \sim U(0,1)$$

Make the program such that the parameters $(N, x, k, \mu, \sigma, \beta, c)$ can be chosen as parameter values. For the concrete calculations, we put

$$(x, k, \mu, \sigma, \beta, c) = (50, 17, 0.055, 0.056, 0.0000202, 1.1015)$$

while the number of insured customers should be able to vary: $N \in \{1, 2, ..., 20\}$.

The asymptotic distribution of P when the number of insured customers go to infinity is given by the expected value. Show that

$$\mathrm{E}(P \mid v_1, v_2, v_3, ...) = \sum_{t=k}^{\infty} v_t \cdot {}_tp_x$$

Compare the standard deviation of the asymptotic distribution, which only considers the financial uncertainty, with the standard deviation in the probability distribution, which considers both demographic uncertainty (life span) and financial uncertainty.

2 Stocks/bonds

We have two stochastic variables,

$$\begin{aligned} S_t &= \text{value of stock at time } t \\ B_t &= \text{value of bond at time } t \end{aligned}$$

The initial values (S_0, B_0) are known, and we assume that the simultaneous probability distribution is given by:

$$\begin{aligned} S_t &= S_0 \exp\left[\left(\mu_S - \frac{\sigma_S^2}{2}\right) t + \sigma_S \cdot V_t\right] \\ B_t &= B_0 \exp\left[\left(\mu_B - \frac{\sigma_B^2}{2}\right) t + \sigma_B \cdot W_t\right] \\ (V_{t_2} - V_{t_1}, W_{t_2} - W_{t_1}) &\sim N\left(\left(\begin{array}{c} 0 \\ 0 \end{array}\right), (t_2 - t_1) \cdot \left(\begin{array}{cc} 1 & \rho \\ \rho & 1 \end{array}\right)\right) \end{aligned}$$

We shall study the stochastic variable for the ratio between the stock price and the bond price at time t

$$Z_t = \frac{S_t}{B_t}$$

Show that

$$Z_t = Z_0 \exp\left[\left(\alpha - \frac{\beta^2}{2}\right) t + \beta \cdot U_t\right]$$

where

$$\begin{aligned} U_t &\sim N(0, \sqrt{t}) \\ \alpha &= (\mu_S - \mu_B) + \sigma_B(\sigma_B - \rho \cdot \sigma_S) \\ \beta^2 &= \sigma_S^2 + \sigma_B^2 - 2 \cdot \sigma_S \cdot \sigma_B \cdot \rho \end{aligned}$$

$\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution. Show that:

$$\Pr\{\frac{Z_t}{Z_0} > k\} = \Phi\left(\left(\alpha - \frac{\beta^2}{2} - \frac{\log(k)}{t}\right) \frac{\sqrt{t}}{\beta}\right)$$

and that:

$$\Pr\{\frac{Z_t}{Z_0} > 1\} > \frac{1}{2} \iff \mu_S - \frac{\sigma_S^2}{2} > \mu_B - \frac{\sigma_B^2}{2}$$

Plot $\Pr\{Z_t > k\}$ and the probability density of Z_t, $f(k)$, for $t \in [0, 40]$ and for the following parameter values:

$$\begin{aligned} S_0 &= 1 \\ B_0 &= 1 \\ \mu_S &= 0.10 \\ \mu_B &= 0.05 \\ \sigma_S &= 0.20 \\ \sigma_B &= 0.05 \\ \rho &= 0.40 \\ k &= 1 \end{aligned}$$

Assume that the share has an expected return i and a standard deviation s per time unit. Show that

$$\begin{aligned} \mu &= \log(1+i) \\ \sigma &= \sqrt{\log\left(1+\left(\frac{s}{1+i}\right)^2\right)} \end{aligned}$$

and illustrate graphically the relationship between i and μ and between s and σ. Show that

$$\mathrm{Cov}\left\{\frac{S_{t+1}-S_t}{S_t}, \frac{B_{t+1}-B_t}{B_t}\right\} = \exp[\mu_S + \mu_B] \cdot (\exp[\rho \cdot \sigma_S \cdot \sigma_B] - 1)$$

and illustrate graphically the relationship between $\mathrm{Corr}\left\{\frac{S_{t+1}-S_t}{S_t}, \frac{B_{t+1}-B_t}{B_t}\right\}$ and ρ.

3 PDF premium reserve

In the model in Exercise 1 we showed that if we take both the financial and the demographic uncertainty into account, the democratic uncertainty will soon drown in the total uncertainty as the number of insured customers increases. In this exercise we shall therefore neglect the demographic uncertainty.

We have a stream of payments $\{G_t,\ t \in [0, T]\}$ which we consider to be deterministic and which is described by the file 'betalingsstrom.txt'. The cash value of this stream of payments as a function of the financial development is

$$K(v_1, v_2, v_3, ...) = \sum_{t=k}^{T} v_t \cdot G_t$$

In Exercise 1, where we studied an annuity, we put $G_t = {}_tp_x$, but this quantity may also include disability benefits and survivor's pensions. In our case, the stream of payments is such that

$$K(v_1, v_2, v_3, ... \mid v_1 = v_2 = v_3 = ... = v) = \sum_{t=k}^{\infty} v^t \cdot G_t \ ,$$

where v is a constant, becomes the premium reserve of a pension scheme.

Using Monte Carlo simulation you shall find a probability distribution for $K(v_1, v_2, v_3, ...)$ under two different models for v_t:

- Normally distributed log-returns

$$\frac{v_{t+1}}{v_t} = \exp\left(-\left(\mu - \frac{\sigma^2}{2}\right) - \sigma \cdot Z_t\right) \ , Z_1, Z_2, ...u.i.d. \sim N(0,1)$$

- Log-returns with t-distributed 'noise terms' with f degrees of freedom and the same variance as the normally distributed log-returns:

$$\frac{v_{t+1}}{v_t} = \exp\left(-\left(\mu - \frac{\sigma^2}{2}\right) - \sigma \cdot \sqrt{\frac{f-2}{f}} \cdot Z_t\right) \ , Z_1, Z_2, ...u.i.d. \sim t(f)$$

In addition to finding the two probability distributions, you shall find which quartiles of the probability distribution $K(v_1, v_2, v_3, ... \mid v_1 = v_2 = v_3 = ... = v)$ corresponds to when we are using the parameter values:

$$\begin{aligned} T &= 80 \\ v &= \frac{1}{1.03} \\ \mu &= 0.055 \\ \sigma &= 0.056 \\ f &= 5 \end{aligned}$$

4 GARCH

GARCH(1,1) is given by

$$\begin{aligned}
\frac{S_{t+1}}{S_t} &= \exp\left(\mu + \sigma_t \cdot Z_t\right)\ , Z_1, Z_2 ... u.i.d. \sim N(0,1) \\
\sigma_{t+1} &= \sqrt{\theta_0 + \theta_1 \cdot (\sigma_t \cdot Z_t)^2 + \theta_2 \cdot \sigma_t^2}
\end{aligned}$$

Show that the variance in the GARCH(1,1)-model has the following properties:

$$\begin{aligned}
\mathrm{E}(\sigma_t^2 \mid \sigma_0^2) &= \theta_0 \cdot \left(\frac{1-(\theta_1+\theta_2)^t}{1-(\theta_1+\theta_2)}\right) + (\theta_1+\theta_2)^t \cdot \sigma_0^2 \\
\lim_{t\to\infty} \mathrm{E}(\sigma_t^2 \mid \sigma_0^2) &= \frac{\theta_0}{1-(\theta_1+\theta_2)}
\end{aligned}$$

Experiment with different parameter values and try to find reasonable values which give noticable accumulations of volatility for periods of as much as 250 consecutive values of the stochastic variables

$$\frac{S_{t+1}}{S_t}\ , t \in \{1, 2, 3, ..., 5000\}$$

when we start with

$$\sigma_0 = k \cdot \sqrt{\frac{\theta_0}{1-(\theta_1+\theta_2)}}$$

A possible procedure is to make plots of

- the log-returns
$$\log\left(\frac{S_{t+1}}{S_t}\right)\ , t \in \{1, 2, 3, ..., 5000\}$$
- the volatilities
$$\sigma_t\ , t \in \{1, 2, 3, ..., 5000\}$$
- and of the "memory"
$$\mathrm{E}(\sigma_t^2 \mid \sigma_0^2)\ , t \in \{1, 2, 3, ..., 250\}$$

A starting point may be the parameter values

$$\begin{aligned}
\mu &= 0 \\
\theta_0 &= 0.000002 \\
\theta_1 &= 0.09 \\
\theta_2 &= 0.89 \\
k &= 3
\end{aligned}$$

5 Annual return

In this exercise we shall look at two models for daily log-returns and study their impact on the annual return. S_t is the value of an asset on day t. and the two models are as follows:

- GBM:

$$\begin{aligned} \frac{S_{t+1}}{S_t} &= \exp\left(\nu + \sigma_0 \cdot Z_t\right)\ , Z_1, Z_2 ... u.i.d. \sim N(0,1) \\ \nu &= \left(\mu - \frac{\sigma_0^2}{2}\right) \end{aligned}$$

- GARCH(1,1):

$$\begin{aligned} \frac{S_{t+1}}{S_t} &= \exp\left(\nu + \sigma_t \cdot Z_t\right)\ , Z_1, Z_2 ... u.i.d. \sim N(0,1) \\ \sigma_0 &= \sqrt{\frac{\theta_0}{1-(\theta_1+\theta_2)}} \\ \sigma_{t+1} &= \sqrt{\theta_0 + \theta_1 \cdot (\sigma_t \cdot Z_t)^2 + \theta_2 \cdot \sigma_t^2} \end{aligned}$$

In the GBM-model we get an explicit expression for probability distribution of S_t:

$$S_t \sim \log N\left(\nu \cdot t, \sigma_0 \cdot \sqrt{t}\right)\ , t \in \{1, 2, 3, ...\}$$

We assume that there are 250 days in a year. We use the same parameters as suggested in Exercise 3 and

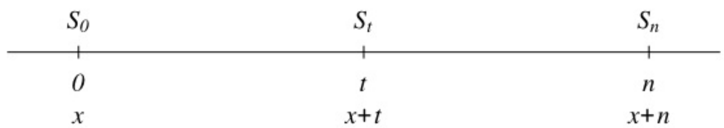
$$\begin{aligned} S_0 &= 1 \\ \nu &= 0 \end{aligned}$$

We wish to see if there are substantial differences in the probability distributions of S_{250} in the two models justifying the use of the more complicated GARCH(1,1)-model. Discuss and compare the difference. Base you analysis on the following properties:

- Graphs showing probability densities
- Expectation
- Variance
- Skewness
- Heaviness of tails

6 Premium development

In this exercise we shall look at a pension amount based scheme with a 'consecutive updating of the pension amount level'.



The first premium is paid at time 0 and equals

$$\frac{1}{n} \cdot S_o \cdot {}_{n|}\ddot{a}_x$$

At any time $t \in \{1, 2, ..., n-1\}$, the premium for increased pension benefits equals:

$$\left(\frac{t+1}{n} \cdot S_t - \frac{t}{n} \cdot S_{t-1}\right) \cdot {}_{n-t|}\ddot{a}_{x+t} = \underbrace{\frac{1}{n} \cdot S_t \cdot {}_{n-t|}\ddot{a}_{x+t}}_{\text{premium in respect of ongoing accrual}} +$$

$$+ \underbrace{\frac{t}{n} \cdot (S_t - S_{t-1}) \cdot {}_{n-t|}\ddot{a}_{x+t}}_{\text{premium in respect of rectroactive pick-up for salary increases}}$$

We have a population consisting of N persons at the age of x years. The pension amount at time t is $S_t = 0.2 \cdot L_t$, where L_t is the actual salary. We assume that everybody in the population has the same salary and the same salary development. The dynamics of the salary development is supposed to follow a stochastic process:

$$L_t = (1 + \lambda) \cdot L_{t-1} + \theta \cdot \delta_t \cdot L_{t-1} \text{ }, \delta_1, \delta_2 ... \delta_n \text{ } u.i.d. \sim N(0, 1)$$

For the concrete computations, we put $N = 1$ as this quantity only contributes as a scaling factor.

- Make a simulation program showing projections of possible trajectories for future premium development, decomposed as 'regular premium' and 'jump premium', assuming that the actual mortality development follows the expected. The basic interest rate is denoted by i. As default values for the parameters, we use the mortality assumptions from Exercise 1 and:

$$\begin{aligned} i &= 0.03 \\ \lambda &= 0.03 \\ \theta &= 0.015 \\ x &= 30 \\ n &= 35 \\ L_0 &= 500000 \end{aligned}$$

Compare to the deterministic special case where $\theta = 0$.

Assume that we have a stochastic return (LN) and that the payments at time t are just sufficient to obtain full coverage of the premium reserve. The necessary premium reserve for the population at time t is:

$$V_t = \frac{t+1}{n} \cdot S_t \cdot {}_{n-t|}\ddot{a}_{x+t} \cdot \underbrace{N \cdot {}_tp_x}_{\text{number}}$$

while the actual develpoment of the premium reserve from t to time $t+1$ is such that the 'premium fund' at time $t+1$ (just before the next payment) equals

$$V_t \cdot (\underbrace{1 + r_{t+1}}_{\text{return}}) \tag{1}$$

where the return from time t to time $t+1$ is given by the stochastic process

$$\begin{aligned} 1 + r_t &= \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) + \sigma \cdot \varepsilon_t\right) \ , \varepsilon_1, \varepsilon_2 ... \varepsilon_n \ u.i.d. \sim N(0,1) \qquad (2) \\ \mu &= 0.055 \\ \sigma &= 0.056 \end{aligned}$$

- Illustrate possible trajectories for the actually necessary payments and compare them to the projected trajectories for future premium development.

7 Premium development with guarantee

In this exercise we build on Exercise 6. In Exercise 6 the return was determined by a stochastic process that could move both above and below the basic interest rate.

To make the model more realistic, we shall assume that the insured are guaranteed that the insurance fund at least follows the basic interest rate, and hence that the actual payments never exceed the premium. The price the insured have to pay for this guarantee, is that they will only receive a part ξ of the return that exceeds the basic interest rate. This guarantee on the return is modeled by replacing

$$1 + r_{t+1}$$

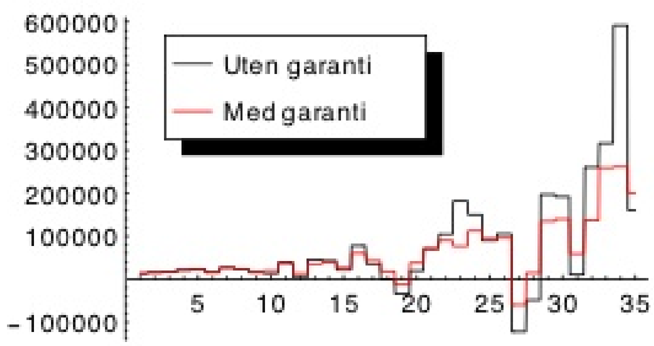
in the equation (1) in exercise 6 by

$$(1 + i) + \xi \cdot (r_{t+1} - i)^{+}$$

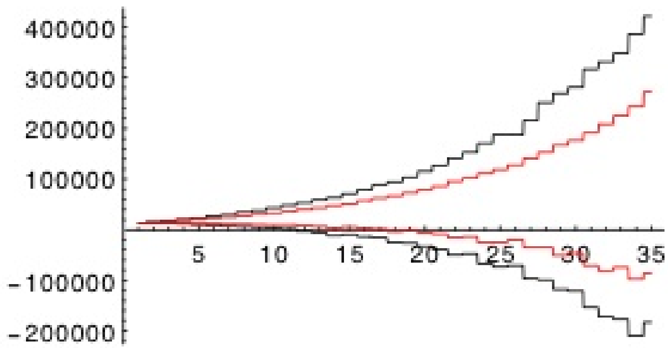
As default value for the new parameter, we use

$$\xi = 0.7$$

- Illustrate possible trajectories for actually necessary payments and compare them to actually necessary payments without a guarantee on the return.

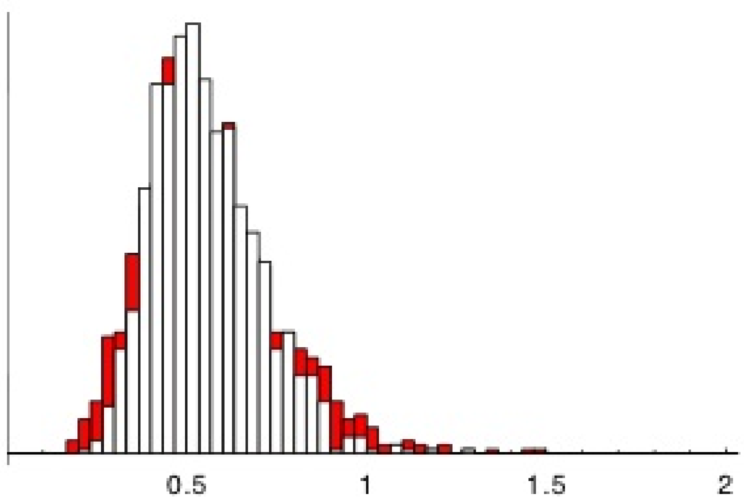


- Increase the number of simulations and make a graphical illustration showing an 'area' where 5% of the simulated payments lie above and 5% lie below at every time t in the future. Do this both with and without a guarantee on the return.



We want to see if the guarantee on the return is profitable for the insured.

- Find the probability distribution of the cash value of the payment stream both with and without the guarantee, when they are discounted by the actual return.



- Find the probability distribution of the ratio between the cash values of the payment streams with and without the guarantee, discounted by the actual return.

8 The Wilkie model

Implement the Wilkie model as described in section 1.4 of the document 'Erik Bølviken ch13.pdf' and study possible trajectories.

9 Deposit based pension

In this exercise we shall take a look at how big we can expect our pension to be if we enter a deposit based pension scheme according to current rules. Assume that the deposits are annual, in advance and of size

$$\begin{aligned} P(L,G) &= 0.05\ \min\{L-2\ G, 4\ G\}\ I(L>2\ G)+ \\ & \quad 0.08\ \min\{L-6\ G, 6\ G\}\ I(L>6\ G) \end{aligned}$$

In the period where deposits are being made, the money is invested in a mutual fund developing according to (2) in Exercise 6. When the retirement age of 67 years is reached, the accumulated capital is invested in an annuity with a return of kroner 13.26 per kroner. The return from the annuity is payed annually and in advance, and will not be adjusted during the disbursement period. To find the total return, we must take the payments from the National Insurance ('folketrygden') into account. For the return from the National Insurance, we use the simplified formula:

$$\begin{aligned} F(L,G) &= 0.825\ G+ \\ & \quad 0.42\ \min\{L-G, 5\ G\}\ I(L>G)+ \\ & \quad \frac{0.42}{3}\ \min\{L-6\ G, 6\ G\}\ I(L>6\ G) \end{aligned}$$

Assume that the salary L and the basic amount of the National Insurance (Folketrygdens grunnbeløp) G are deterministic and develop according to

$$\begin{aligned} G_t &= G_0 \cdot (1+g_G)^t,\ G_0 = 58778 \\ L_t &= L_0 \cdot (1+g_L)^t \end{aligned}$$

Find the probability distribution of the total pension in percent of salary at retirement and later in the disbursement period. You must choose parameters for annual adjustments in salary and G, as well as an initial salary L_0 and the age at which the first deposit is made. In the disbursement period, the pension is to be measured in percent of an assumed, G-adjusted salary.

10 Transferring to deposit based pension

In this exercise we shall compare the pension level and the costs of continuing a return based pension (YP) versus transferring to a deposit based pension scheme (IP). To make the pension levels comparable in the two alternatives, we assume that the deposit account is used to buy an annuity when the retirement age of 67 years is reached. In contrast to Exercise 9 we shall in this exercise compute in continuous time, and we shall assume that the returns are deterministic. We shall use a pure Gompertz death rate (see the Exercise 1) with parameters $(\beta, c) = (0.0000014, 1.14)$. In addition, we have the following list of parameters and functions::

Parameter	Default	Forklaring
x	40	age at entering
t_{ov}	10	length of service at the time of transfer
L	400000	salary
G	58778	the basic amount of the National Insurance
i	0.03	the discount rate of the basis for calculation
m	0.01	margin/ price for the return guarantee in YP
δ		the annual interest intensity in the basis for calc., m incl.
a		actual annual return
δ'		assumed actual annual interest intensity
g	0.03	annual increase in return, L og G
γ		intensity of annual fincrease in return, L og G
S_{YP}		annual return YP
p_{YP}	0.66	pension percentage YP
F		calculated benefits from National Insurance
V_t		premium reserve of YP at time t
π_t		premium intensity at time t
S_{IP}		annual return IP
P		deposit as in exercise 9
E_{67}	13.26	cash value factor for from age 67

We have the following dependent quantities:

$$\begin{aligned} n &= 67 - x \\ \gamma &= \log(1+g) \\ \delta' &= \log(1+a) \\ \delta &= \log(1+i+m) \\ v &= \frac{1+g}{1+a} \end{aligned}$$

Show that

$$S_{IP} \cdot E_{67} = \begin{cases} P \cdot (1+a)^{n-t_{ov}} \cdot \frac{v^{n-t_{ov}}-1}{\log(v)} & , v \neq 1 \\ P \cdot (1+a)^{n-t_{ov}} \cdot (n-t_{ov}) & , v = 1 \end{cases}$$

YP return is computed by the following formulas

$$
\begin{aligned}
S_{YP} &= \min\left\{1, \frac{[n]}{30}\right\} \cdot (p_{YP} \cdot \min\{12 \cdot G, L\} - F)^+ \\
F &= 0.75 \cdot G + 0.42 \cdot \min\{L - G, 5 \cdot G\}^+ + \frac{0.42}{3} \cdot \min\{L - 6 \cdot G, 6 \cdot G\}^+
\end{aligned}
$$

In the alternative of a transfer to IP at time t_{ov}, we will get a so-called ‘paid-up policy’ (fripolise) from the YP. The value of the paid-up policy at retirement age is

$$\frac{t_{ov}}{n} \cdot S_{YP} \cdot (1 + \max\{0, \min\{g, a - i - m\}\}^{n-t_{ov}}$$

The premium intensity of YP is defined as the contribution necessary to obtain linear accumulation, i.e.

$$\frac{dV_t}{dt} = \pi_t + \frac{(\mu_{x+t} + \max\{\delta, \delta^{'}\}) \cdot t}{n} \cdot S_t \cdot E_{x+t}$$

Show that

$$\pi_t = \frac{1 + (\gamma - (\delta^{'} - \delta)^+) \cdot t}{n} \cdot S_t \cdot E_{x+t}$$

Finally, you shall implement the necessary quantities and, under varying assumptions on actual return $a \in (0.00, 0.10)$, compare

- S_{YP} (when $p_{YP} \in \{0.66, 0.60\}$) and S_{IP} in percent of salary at retirement
- P og π (når $p_{YP} = 0.66$) in percent of salary from today till retirement.

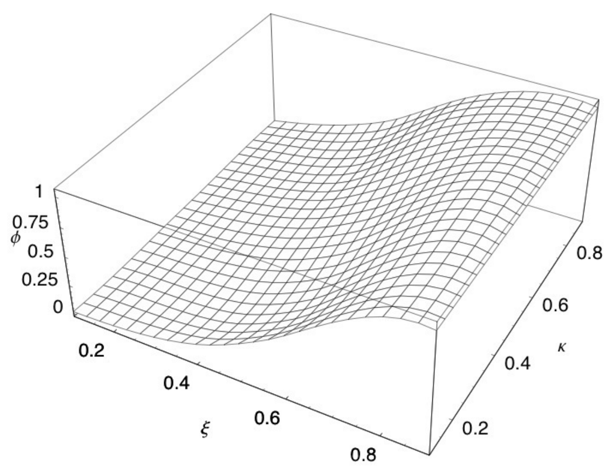
11 Profit sharing

We start with the model and the parameters from Exercise 9.

We let equation (2) describe the development of the value of a portfolio consisting of stocks and bonds developing according to the model in Exercise 2. The portfolio is rebalanced every year such that the value of the part of the portfolio that is in stocks is kept constant. We also make the assumption that a weighted sum of our lognormal stochastic variables is approximately lognormal. Show that the parameters we used in equation (2) in Exercise 9 corresponds to having $\kappa = 0.10$ of the portfolio in stocks. In the rest of the exercise, the parameters (μ, σ) will be functions of the part κ of the portfolio that is in stocks.

The other new thing in this exercise is that we have a guarantee on return that is priced implicitly through profit sharing. The insurance company guarantees that the deposit account pays the basic interest of 3% every year, but in return the account is only accredited a part ξ of the return exceeding the basic interest rate. For the insurance company this scheme includes a possibility for loss depending on the return. The 'fair' part ξ_0 can be determined such that the probability for loss is less than ϕ

- Find the probability distribution of the total pension in percent of the salary at retirement with and without the guarantee. Compare!
- Make a 3D-plot showing ξ_0 as a function of κ and ϕ



12 Pricing guarantees on return

Guarantees on return can also be priced explicitly as European put options as described in the paper 'Pricing of minimum interest guarantees: Is the arbitrage free price fair?'. This exercise is based on the model and the notation of the paper. We use the parameters from the section '4 Case Study' as default values.

- Make a calculator for a one period return guarantee based on equation (10) in the paper.The calculator should give error messages for parameter values that do not lead to a solution (see appendix A in the paper).
- Make the calculations needed to produce Figure 4 in the paper

The annual return guarantee can be met by either buying a put option or by buying the replicating portfolio of the put option. For the replicating portfolio at any time to have exactly the same value as the put option, it has to be continuously rebalanced. In practice, the portfolio must be rebalanced slightly delayed and at discrete times. Assume that we divide the year in a certain number of trading days where the portfolio can be rebalanced. On trading day $t+1$ we rebalance the portfolio such that if we had this portfolio on trading day t, the value on trading day $t+1$ would had been the same as the value of the put option.

- Make a plot of the replicating portfolio which corresponds to the annual return guarantee in the paper.

  $$\left\{a_t \cdot S_t + b_t \cdot \exp\left(-\delta \cdot \left(1 - \frac{t}{h}\right)\right)\right\}_{t=0,\ldots,h-1}$$

  where

  $$a_t = -\Phi\left(-\left(\frac{\log\left(\frac{S_t}{K}\right) + \left(\delta + \frac{\sigma^2}{2}\right) \cdot \left(1 - \frac{t}{h}\right)}{\sigma \cdot \sqrt{1 - \frac{t}{h}}}\right)\right)$$

  and $h \in \{250, 50, 12\}$.

- In continuous time the replicating portfolio is self-financing, but the delayed rebalancing in discrete time needs added capital. Find the distribution of cash value of the added capital divided by the option price.

13 Project

In this exercise we shall do the same as in exercises 6 and 7, but based on the Wilkie model from exercise 8. We look at a closed group of people of the same age. They have the same salary development determined by

$$\begin{aligned} L_k &= L_{k-1}\,(1+I_k^l) \\ I_k^l &= I_k + l + Z_k^l \\ Z_k^l &= \sigma^l\,\varepsilon_k^l,\ \varepsilon_k^l \sim N(0,1) \end{aligned}$$

where

$$\begin{aligned} l &= 0.015 \\ \sigma^l &= 0.01 \end{aligned}$$

The return is determined by the development of the value of a portfolio of shares and bonds following the Wilkie model from exercise 8. Note that we have no use for inflation in this exercise as it is included as part of the model generating stocks, interest rates, and salary. Note also that the model for salary development includes I_k from the model for inflation. This quantity must not be confused with I_k^l.

14 Computation of efficient set and portfolio front.

We look at a market with three (risky) investment alternatives characterized by:

$$\mu = \begin{pmatrix} 0.06 \\ 0.08 \\ 0.11 \end{pmatrix}$$

and

$$V = \begin{pmatrix} 0.0025 & -0.002 & 0.003 \\ -0.002 & 0.01 & 0.01 \\ 0.003 & 0.01 & 0.04 \end{pmatrix}$$

Compute x^{MIN} and z^*.

Illustrate the portfolio front graphically in the $\left(r, \sigma^2\right)$-plane as well as in the (r, σ)-plane.

15 Optimization of investment portfolios with risk free investment alternatives.

In this exercise we look at a situation where the investment universe is extended by a risk free investment alternative. For the n risky investment alternatives, we are using the same concepts and notation as before:

$R_i =$ Value at the end of the period of unit invested in property $i; i = 1, ..., n$
$\mu_i = \mathrm{E}(R_i); i = 1, ..., n$
$V_{ij} = \mathrm{Cov}(R_i, R_j); i, j = 1, ..., n$
$\mu =^t (\mu_1, ..., \mu_n)$
$V = \{V_{ij}\}_{i,j=1,...,n}$

We denote the risk free alternative as "alternative 0", and let R_0 denote the (deterministic) value at the end of the period of one unit invested in this alternative.

As in the case without a risk free alternative, we look at optimization of the investment portfolio in the following way: Given 1) one unit at our disposal for making investments, and 2) a chosen expected level of return, r, we shall put together a portfolio of the $(n+1)$ investment alternatives in such a way that the variance of the value at the end of the period is as small as possible.

Formulate the optimization problem in a precise mathematical way, and findthe Lagrange function needed to solve the problem.

15.1

For the optimal parts invested in properties 0 and $1, ..., n$, respectively, we introduce the notation $\widehat{x}_0$ and $\widehat{x} =^t (\widehat{x}_1, ..., \widehat{x}_n)$. Show that $\widehat{x}_0$, $\widehat{x}$ and the Lagrange multipliers 2λ and 2ν must satisfy:

1) $V\widehat{x} - \lambda e - \nu\mu = 0$
2) $\lambda + \nu R_0 = 0$
3) $\widehat{x}_0 R_0 +^t \widehat{x}\mu = r$
4) $\widehat{x}_0 +^t \widehat{x}e = 1$

where we have used the notation $e =^t (1, ..., 1) \in \mathbb{R}^n$.

15.2

Show that $\widehat{x}$ must satisfy:

$V\widehat{x} = \nu(\mu - R_0 e)$

15.3

Show that:

$var\left(R\left(\widehat{x}\right)\right) = \nu\left(r - R_0\right)$

and:

$\text{Cov}\left(R_i, R\left(\widehat{x}\right)\right) = \nu\left(\mu_i - R_0\right); i = 1, ..., n$

15.4

Show that:

$$\left(\mu_i - R_0\right) = \frac{\text{Cov}\left(R_i, R\left(\widehat{x}\right)\right)}{var\left(R\left(\widehat{x}\right)\right)}\left(r - R_0\right); i = 1, ..., n$$

and express in words what this relation tell us.

This is an essential relationship in CAPM (Capital Asset Pricing Model). One often refers to $\frac{\text{Cov}(R_i, R(\widehat{x}))}{var(R(\widehat{x}))}$ as the security's β, and it may be interpreted as an expression for how sensitive the security is compared to the total market. .

16 Mathematical properties at the portfolio front.

Let:

$$\sigma \left\{ R \left[x^* \left(r \right) \right] \right\}$$

denote the standard deviation for the efficient set corresponding to a chosen return level r. Derive an analytic expression for the tangent through the point

$$\left(r_0, \sigma \left\{ R \left[x^* \left(r_0 \right) \right] \right\} \right)$$

at the portfolio front. Show that the portfolio front has an asymptote which passes through the point

$$\left(\mathrm{E} \left\{ R \left(x^{MIN} \right) \right\}, 0 \right)$$

and has slope

$$\frac{\sigma \left\{ R \left[z^* \right] \right\}}{\mathrm{E} \left\{ R \left[z^* \right] \right\}}$$

Comment.

17 "Free lunch" with a barbell strategy

In this exercise we look at a portfolio combining zero coupon bonds of different maturity. By a zero coupon bond we mean a bond which only pays a unit at maturity. It is possible to be "short" in a bond, i.e. "to borrow" a bond/oblige oneself to pay the unit back at maturity.

We consider the construction of the portfolio as follows: We lend a bond of medium maturity m. The money received is invested in parts of two different bonds, one with a short time s to maturity and one with a long time l, i.e. $s < m < l$. Assume that the interest rate is constant through the period – denote it by $r = e^{\rho} - 1$ – and that the possible variations in the interest are horizontal shifts in the constant interest intensity.

17.1

Show how to determine a self-financing and interest immune portfolio.

17.2

Find analytic expressions for the parts that should be bought in the long and the short term bond, respectively.

17.3

Show that the profit, i.e. the cash value of the cash flow from the obligations bought subtracted the cash flow of the bond "shorted", has a local minimum at r.

17.4

To make concrete calculations, we put $r = 0.03, s = 5, m = 10, l = 15$. Calculate how the portfolio is constructed and illustrate the profit as a function of the interest rate.

17.5

Comment.