Pricing of minimum interest guarantees: Is the arbitrage free price fair?

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Abstract

Defined contribution (DC) pension schemes and life insurance contracts often have a minimum interest guarantee as an integrated part of the contract. The purpose of this paper is to study the "value" to the policyholder of an embedded interest rate guarantee, depending on how the guarantee is priced. We study a simple savings DC plan where the minimum interest guarantee is expressed on an annual basis, and it is assumed that the guarantee is priced according to the arbitrage free principle. We use stochastic simulation to develop the (approximate) probability distributions for the amount on the DC account at retirement - respectively with and without a minimum interest rate guarantee embedded. The probabilistic properties of the two alternative scenarios indicate that the "enhanced safety" achieved by the guarantee is small compared to the loss of potential return implied by charges for providing the guarantee. On this basis we raise the question of whether an "equilibrium price" for the interest rate guarantee can be established.

1 Introduction

Minimum interest guarantees have always been an implicit part of ordinary pension schemes and life insurance contracts. They have traditionally not been priced separately from the rest of the premium. From an actuarial point of view it is reasonable that they should have a price, which we can consider as a risk premium against an adverse development in the financial markets. However, it is not obvious what this risk premium should be.

In contrast to demographic risk financial risk is not reduced when the number of contracts in the insurance portfolio is increased. Therefore the law of large numbers can not be applied to price the guarantee. Following the route of option pricing theory from mathematical finance we can employ alternative ways to reduce and theoretically in fact completely eliminate the risk. Within the framework of this theory we can find premiums which correspond to a complete absence of risk for the provider.

In this paper we first specify a general model for the financial market where there exists one and only one arbitrage free guarantee price. Then we apply this model to a certain long term savings plan and develop the formulae needed for the actual calculations of the savings plan dynamics. A numerical case study illustrates some properties of our model. At the end of this paper we discuss some possible interpretations and implications of our results.

2 General model

We use the standard Black&Scholes setting, where we assume that there exists two assets in the market, a bond with a deterministic force of interest δ and a stock with a stochastic Gaussian log-return.

- A bond with current value B_0 has a value at time t:

$$B_t = B_0 \, e^{\delta t} \tag{1}$$

- A stock with current value S_0 has a value at time t :

$$S_t = S_0 \, e^{L_t} \tag{2}$$

  where the log-return is $L_t \sim N(\nu t, \sigma\sqrt{t})$.

The parameter ν can be evaluated from an assumption on the stock's expected rate of return until time t. If we assume that the expected return corresponds to a force of interest μ, we have

$$\begin{aligned} E[S_t] &= S_0 \, e^{\mu t} \Leftrightarrow \\ S_0 \, e^{\left(\nu t + \frac{\sigma^2 t}{2}\right)} &= S_0 \, e^{\mu t} \Leftrightarrow \\ \nu &= \mu - \frac{\sigma^2}{2} \end{aligned} \tag{3}$$

Let us assume that we have a minimum interest rate guarantee on an investment in the stock at time m based on a given minimum force of interest γ. Having this guarantee is the same as having the right to sell the stock for $S_0 \, e^{\gamma m}$ if the value of the stock is below this level. This guarantee is actually a put option with strike price $K = S_0 \, e^{\gamma m}$ at maturity m. The value of a put option at time m is $(K - S_m)^+$ and the arbitrage free price is

$$\begin{aligned} p_0 &= e^{-\delta m} \, E_Q[(K - S_m)^+] \, , Q \sim N(\delta m - \frac{\sigma^2 m}{2}, \sigma\sqrt{m}) \\ &= K \, e^{-\delta m} \, \Phi(-d_2) - S_0 \, \Phi(-d_1) \end{aligned} \tag{4}$$

$$\begin{aligned} d_1 &= \frac{\log(\frac{S_0}{K}) + (\delta m + \frac{\sigma^2 m}{2})}{\sigma\sqrt{m}} \\ d_2 &= d_1 - \sigma\sqrt{m} \end{aligned}$$

3 The savings account

We consider a savings plan where contributions are made annually in advance. The savings account is invested in the two assets bonds and stocks specified for the financial market. The bond/ stock proportion is assumed to be stable over time, which is achieved by a rebalancing approach. Formally we are looking

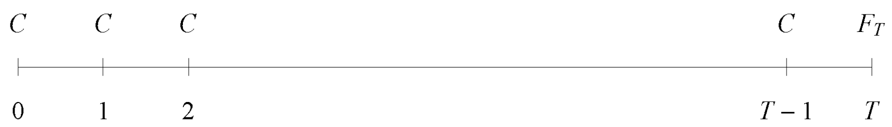

Figure 1: The savings account.

at a savings account $\{F_t\}_{t=0,1,2,...T}$ described in Figure 1. We assume that we have annual payments C in advance and define a function, a_t, that determines the value at time t of a unit invested at time $t-1$:

$$a_t = \alpha\, e^{G_t} + (1-\alpha)\, e^{\delta} \tag{5}$$

Here $G_t = L_t - L_{t-1} \sim N(\nu, \sigma)$. $\alpha \in (0,1)$ is the share/ weight invested in a given stock which develops according to (2) and $(1-\alpha)$ is the share/ weight invested in a bond which develops according to (1). It is worth mentioning that we could easily generalize this model to deterministic time dependent annual payments. We could then simply substitute C with C_t in all the following formulae.

3.1 The savings account without guarantee

The savings account **without** guarantee develops according to the recursion formula

$$\begin{aligned} F_0 &= 0 \\ F_t &= a_t\,(C + F_{t-1})\,, t = 1, 2, ...T \end{aligned} \tag{6}$$

3.2 The savings account with guarantee

In order to establish a comparative basis to (6) for the dynamics of the account **with** guarantee, we need to consider carefully how the guarantee is provided and paid for. For illustrative purposes we will first describe a simplified approach, before we present the model which we will actually adopt.

3.2.1 The simple model

We start by considering a model where the guarantee premiums are paid in addition to C at the beginning of each year. Under this model the savings account **with** guarantee would develop according to the recursion formula

$$\begin{aligned} F_0^g &= 0 \\ F_t^g &= \max\{e^\gamma, a_t\}\ (C + F_{t-1}^g)\ , t = 1, 2, ...T \end{aligned} \tag{7}$$

If the guarantee has a potential value, the unit arbitrage free price in this model for a guarantee the following year is

$$\begin{aligned} p_1 &= e^{-\delta}\ E_Q[(e^\gamma - a_t)^+]\ , Q \sim N(\delta - \frac{\sigma^2}{2}, \sigma) \\ &= e^{-\delta}\ E_Q[(e^\gamma - (1-\alpha)\ e^\delta - \alpha\ e^{G_t})^+] \\ &= e^{-\delta}\ E_Q[(K - S_0\ e^{G_t})^+] \\ &= K\ e^{-\delta}\Phi(-d_2) - S_0\ \Phi(-d_1) \end{aligned} \tag{8}$$

$$\begin{aligned} d_1 &= \frac{\log(\frac{S_0}{K}) + (\delta + \frac{\sigma^2}{2})}{\sigma} \\ d_2 &= d_1 - \sigma \\ K &= e^\gamma - (1-\alpha)\ e^\delta \\ S_0 &= \alpha \end{aligned}$$

We see that the guarantee has a potential value if $K > 0$, that is if the amount guaranteed, e^γ, is higher than the value of the risk free investment, $(1-\alpha)\ e^\delta$.

The problem with this approach to modelling the dynamics of the savings account **with** guarantee is that the payment in year t is $C + p_1\ (C + F_t^g)$ and not just C as for the savings account **without** guarantee. The cash flow in the two cases are different in two ways: **With** guarantee we firstly have to pay something in addition to the ordinary contribution and secondly this additional amount is a stochastic variable. Because of this difference in cash flows from the policyholder's point of view, it is not obvious how the two approaches can be compared.

3.2.2 Our model

In order to have a comparative basis from the policyholder's point of view, we assume that the guarantee-premium is charged to the savings account at the beginning of each year. In doing so we have to take into account that the actual return will be reduced after deduction of the guarantee premium and that the guarantee premium itself should not be included in the amount for which we need a guarantee. The dynamics of the savings account can in this case be evaluated from

$$F_t^g = \max\{e^\gamma, a_t\ (1-p)\}\ (C + F_{t-1}^g) \tag{9}$$

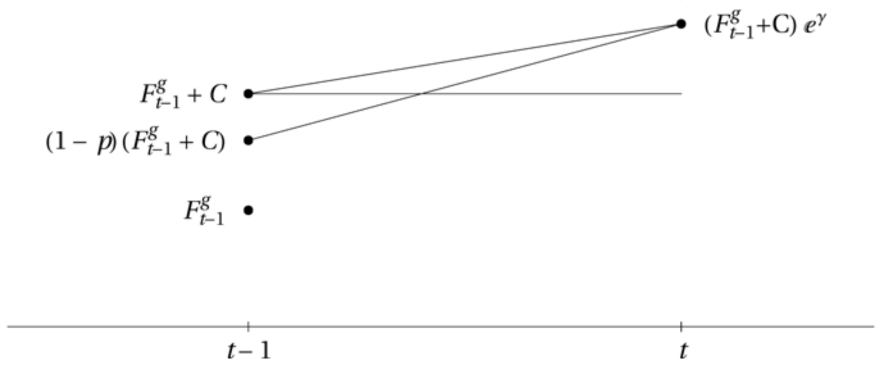

Figure 2: The dynamics of the guarantee from time $t-1$ to time t.

where p is the unit guarantee premium. The dynamics of this savings account is shown in Figure 2. The unit guarantee premium p is obtained as the solution of the equation

$$\begin{aligned} p &= e^{-\delta}\, E_Q[(e^\gamma - (1-p)\, a_t)^+] \\ &= e^{-\delta}\, E_Q[(e^\gamma - (1-p)\,(1-\alpha)\, e^\delta - (1-p)\,\alpha\, e^{G_t})^+] \\ &= e^{-\delta}\, E_Q[(K - S_0\, e^{G_t})^+] \\ &= K\, e^{-\delta}\Phi(-d_2) - S_0\,\Phi(-d_1) \end{aligned} \tag{10}$$

$$\begin{aligned} d_1 &= \frac{\log(\frac{S_0}{K}) + (\delta + \frac{\sigma^2}{2})}{\sigma} \\ d_2 &= d_1 - \sigma \\ K &= e^\gamma - (1-p)\,(1-\alpha)\, e^\delta \\ S_0 &= (1-p)\,\alpha \end{aligned}$$

In (10) p appears both at the left hand side and the right hand side of the equation. In the absence of an explicit analytical solution, a numerical approach has to be employed. In this case the guarantee premium has a potential value if the amount guaranteed, e^γ, is higher than the value of the remaining risk free investment after subtraction of the guarantee premium, that is $(1-p)\,(1-\alpha)\,e^\delta$.

To compare F_T and F_T^g we define a stochastic variable, Ψ_T, which we might call pension enhancement

$$\Psi_T = 100\left(\frac{F_T^g}{F_T} - 1\right) \tag{11}$$

This variable tells us how many percent higher the final amount on the savings account gets with guarantee than without.

3.3 Comparing the savings accounts

We assume that it is difficult or impossible to find an explicit expression for the cumulative probability distribution or the probability densities for F_T and F_T^g and therefore we use stochastic simulation to examine their probabilistic properties. First we simulate a sequence of Gaussian log-returns $G_1, G_2, ..., G_T$. For one particular realization of this sequence we can calculate the corresponding F_T and F_T^g (and Ψ_T) according to the formulae (6) and (9). This algoritm is then repeated n times and we end up with n simulated realizations of F_T and F_T^g (and Ψ_T). If n is big enough, the realizations of these three stochastic variables will be distributed according to their underlying probability density.

Because the purpose of the guarantee is to increase the financial safety for the policy holder, we are particularly interested in worst case scenarios. To compare the lower tails of the probability densities of F_T and F_T^g we consider the measures Value at Risk (VaR) and Conditional Value at Risk ($CVaR$) which for a stochastic variable X are defined as

$$VaR(\varepsilon) = q_\varepsilon \tag{12}$$

$$CVaR(\varepsilon) = E[X \mid X < q_\varepsilon] \tag{13}$$

where

$$\Pr\{X < q_\varepsilon\} = \varepsilon$$

For n stochastic realizations $x_1, x_2, ..., x_n$ of X an estimate of $VaR(\varepsilon)$ is given by $x_{(\varepsilon n)}$ and an estimate of $CVaR(\varepsilon)$ is given by

$$\frac{\sum_{i=1}^{n} x_i I(x_i < x_{(\varepsilon n)})}{\sum_{i=1}^{n} I(x_i < x_{(\varepsilon n)})} \tag{14}$$

where $x_{(1)} < x_{(2)} ... < x_{(n)}$ are the ordered realizations of X and n is such that the product εn is an integer.

γ is the force of interest which corresponds to the minimum rate of return "from the policyholder's point of view". We are also interested in the force of interest on the reduced account $(1-p)\ (C + F_{t-1}^g)$, called γ^a, which corresponds to the minimum rate of return "from the provider's point of view". From Figure 2 we see that it is given by the equation

$$(1-p)\ e^{\gamma^a} = e^{\gamma} \Leftrightarrow$$

$$\gamma^a = \gamma - \log(1-p) \tag{15}$$

4 Case study

We have now completely specified the savings plan under the two different approaches. In this section we are going to put some concrete numbers into the variables to illustrate the properties of the model. We assume that $C = 1$ and that

$$\begin{aligned} T &= 20 \text{ years} \\ \sigma &= 20\ \% \text{ per year} \\ \mu &= 10\ \% \text{ per year} \\ \delta &= 5\ \% \text{ per year} \\ \gamma &= 3\ \% \text{ per year} \\ \alpha &= 20\ \% \end{aligned}$$

This is supposed to be a reasonably representative description of long term properties of a typical financial market and of typical terms of DC contracts.

The force of interest for the minimum guarantee as perceived by the provider is $\gamma^a = 4.1\%$ and the unit guarantee premium is $p = 1.1\%$.With $n = 10^5$ simulations of each of F_T and F_T^g we obtain the estimated probability densities for F_T and F_T^g shown in Figure 3. The minimum values, the estimated Value at Risk and the estimated Conditional Value at Risk are shown in the data summary in Table 1.

	min	$VaR(.05)$	$CVaR(.05)$
F_T	26.4	32.7	31.4
F_T^g	29.3	33.1	32.3

Table 1: Data summary

The estimated probability density for Ψ_T is shown in Figure 4. Only 20% of the probability mass lies to the right side of 0 in this figure. It is noteworthy and remarkable that with a probability of 80% the policyholder is best served by abstaining from the guarantee.

We have also studied the sensitivity of $\Pr\{\Psi_T > 0\}$ to changes in the parameters μ and σ in Table 2:

		σ		
		.10	.20	.30
	.07	.26	.37	.46
μ	.10	.09	.20	.30
	.15	.01	.05	.12

Table 2: $\Pr\{\Psi_T > 0\}$

As we might expect this probability is increasing in σ and decreasing in μ. This supports the following intuitive insights:

- "The riskier the financial market, the more likely it is that the guarantee will be of benefit".
- "The better expected performance in the financial market, the less likely it is that the guarantee will be of benefit."

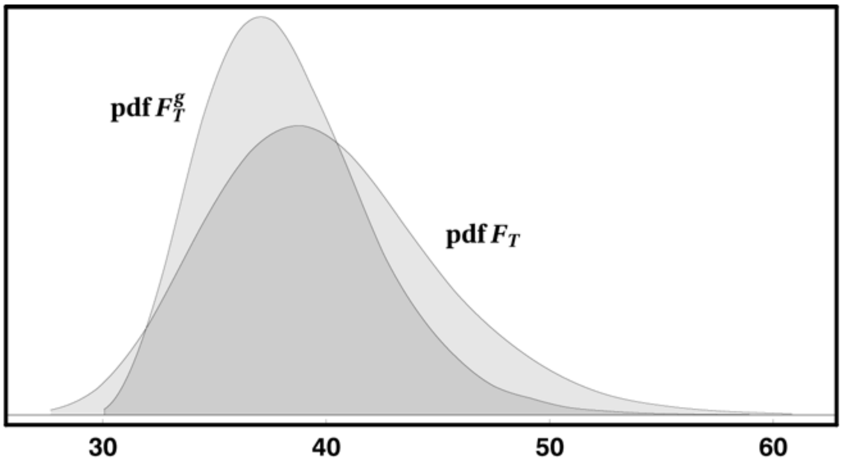

Figure 3: Approximate probability densities for F_T and F_T^g.

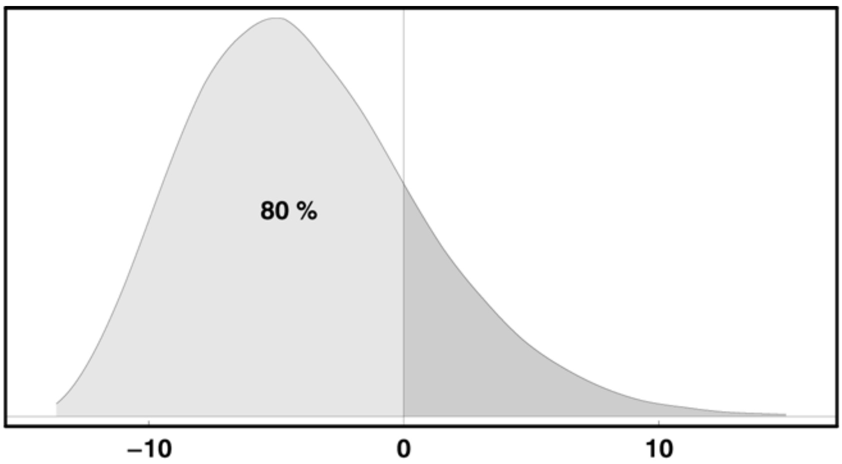

Figure 4: Approximate probability density for Ψ_T.

5 Conclusion

The concept of minimum interest rate guarantees is well established within actuarial theory and practice dealing with traditional life insurance savings policies. However, from a contractual standpoint the interest rate guarantee itself is not highlighted; rather it is a means towards fulfilling the obligations of paying fixed benefits under the contracts.

All other things being equal, it is obvious that a savings contract with an embedded interest rate guarantee is more valuable than a similar policy without the guarantee. Accordingly the interest rate guarantee should have a price. To the authors it is however unclear to what extent and eventually how life insurance providers have charged policyholders for such a price. Traditional actuarial theory disregards the cost and the pricing of interest rate guarantees.

For policyholders and beneficiaries an interest rate guarantee provides a shelter from the financial market's downside risk. On the surface this appears to be a user-friendly concept also for unit-linked (UL) and DC contracts.

However, at this point there is one important distinction between traditional contracts and UL/DC-contracts to be taken into consideration, in that for the latter the provider does not exercise control over the asset allocation as a means towards achieving a financial performance embedded in the guarantee. Accordingly, the question of putting an explicit price on the guarantee comes to the forefront.

Our main focus of this paper has been to investigate whether there is a reasonable balance between this price on the one hand and the corresponding enhanced safety on the other hand.

An interest rate guarantee for a certain investment portfolio is recognized as a put option, and generally accepted theory of financial risk provides us with the answer of how this option should be priced. Underlying the theory is that the provider exercises a certain risk-neutralizing investment behavior, which will enable him to deliver exactly the contractual return on the account irrespective of the actual performance of the financial market.

Within the context of a UL/DC-account, the perspective of charging an explicit premium for the interest rate guarantee based on a risk-neutralizing single period perspective meets the dynamics of the account in a multi-period perspective. While there is no diversification underlying the single period pricing of the guarantee - here risk is completely eliminated as opposed to diversified - , a certain "time diversification" effect is achieved when we consider the development of the savings account over time. Accordingly it is meaningful to measure the "fairness" of the arbitrage-free risk premium in probabilistic terms, as demonstrated in this paper by the probability distributions of the terminal account values.

It is a general impression that the safety the policyholder achieves from the interest rate guarantee is small compared to the reduced return resulting from the guarantee premium charges. Considered as an insurance coverage, the insurance premium appears to be excessively priced. Moreover, an enhanced safety which is broadly similar, can be achieved by shifting towards a less risky asset allocation.

Underlying this result is that the guarantee premium is systematically higher than its expected payoff. This is because the risk-neutral Q-measure which we use to derive the guarantee premium is located to the left of the true probability measure (for the problem to be of real interest, we must have $\mu > \delta$) and it is stock return realizations in this "left region" that contribute to the expected value-calculation.

Philosophically, it appears to be too expensive to allow the provider to do away with all risk as a basis for providing the guarantee. However, this should not be confused with greedy, over-pricing behavior by the provider. In fact, the role of the provider can be eliminated from our model. Instead we could assume that the policyholder went directly with his proceeds to the financial market and purchased the portfolio which replicated the interest rate guaranteed account. We would then obtain exactly the same results.

In applying interest rate guarantees to UL/DC-contracts the following paradox arises:

- In order to avoid arbitrage, the risk-neutral approach represents the only solution to how the premium should be priced
- Applying the risk-neutral premium over the duration of the contract is broadly to the disadvantage of the policyholder.

On this background, the authors question whether a meaningful equilibrium premium for an interest guarantee applied to UL/DC-contracts can be established in the market.

Finally, we should not disregard the real protective properties against worst case scenarios that the interest rate guarantee *does* have. This is demonstrated in Table 1, which shows both the absolute lowest outcome from the simulation, the VaR-threshold and the CVaR-value are enhanced by including the guarantee.

A Solving the equation (10)

We define $f(x)$ which we assume to be a smooth function $\forall\, x \in [0,1]$:

$$\begin{aligned}
f(x) &= x - (K\ e^{-\delta}\ \Phi(-d_2) - S_0\ \Phi(-d_1)) \\
\\
K(x) &= e^{\gamma} - (1-x)\ (1-\alpha)\ e^{\delta} \\
S_0(x) &= (1-x)\ \alpha \\
\\
d_1(x) &= \frac{\log(\frac{S_0}{K}) + (\delta + \frac{\sigma^2}{2})}{\sigma} \\
d_2(x) &= d_1 - \sigma
\end{aligned}$$

We show that the first derivative of f is strictly positive:

$$\begin{aligned}
K'(x) &= (1-\alpha)\ e^{\delta} \\
S_0'(x) &= -\alpha \\
\\
d_2'(x) &= -\frac{\alpha\ e^{\gamma}}{S_0\ K\ \sigma} \\
d_1'(x) &= d_2(x) \\
\\
\Phi'(x) &= \phi(x) \\
\phi'(x) &= -x\ \phi(x) \\
\\
\phi(d_2) &= \phi(d_1)\ \left(\frac{S_0}{K}\right)\ e^{\delta} \\
\\
f'(x) &= 1 - \left((1-\alpha)\ \Phi(-d_2) + \phi(-d_2)\ \frac{\alpha\ e^{\gamma-\delta}}{S_0\ \sigma}\right) + \\
&\quad \left(-\alpha\ \Phi(-d_1) + \phi(-d_1)\ \frac{\alpha\ e^{\gamma}}{K\ \sigma}\right) \\
&= 1 - (1-\alpha)\ \Phi(-d_2) - \alpha\ \Phi(-d_1) - \frac{\alpha\ e^{\gamma}}{\sigma}\left(\frac{e^{-\delta}}{S_0}\ \phi(d_2) - \frac{1}{K}\ \phi(d_1)\right) \\
&= 1 - (1-\alpha)\ \Phi(-d_2) - \alpha\ \Phi(-d_1) - \\
&\quad \frac{\alpha\ e^{\gamma}}{\sigma}\left(\frac{e^{-\delta}}{S_0}\ \phi(d_1)\ \left(\frac{S_0}{K}\right)\ e^{\delta} - \frac{1}{K}\ \phi(d_1)\right) \\
&= 1 - (1-\alpha)\ \Phi(-d_2) - \alpha\ \Phi(-d_1) \\
\\
f'(x) &> 1 - (1-\alpha) - \alpha = 0\ \forall\, x \in (0,1)
\end{aligned}$$

We find the second derivative of $f(x)$:

$$\begin{aligned}
f''(x) &= -(1-\alpha)\,\phi(-d_2)\,\frac{\alpha\,e^{\gamma}}{S_0\,K\,\sigma} - \alpha\,\phi(-d_1)\,\frac{\alpha\,e^{\gamma}}{S_0\,K\,\sigma} \\
&= -\frac{\alpha\,e^{\gamma}}{S_0\,K\,\sigma}\,((1-\alpha)\,\phi(d_2) + \alpha\,\phi(d_1)) \\
&= -\frac{\alpha\,e^{\gamma}}{S_0\,K\,\sigma}\left((1-\alpha)\,\phi(d_1)\,\left(\frac{S_0}{K}\right)\,e^{\delta} + \alpha\,\phi(d_1)\right) \\
&= -\frac{\alpha\,e^{\gamma}\,\phi(d_1)}{S_0\,K\,\sigma}\left((1-\alpha)\,\left(\frac{S_0}{K}\right)\,e^{\delta} + \alpha\right) \\
&= -\frac{\alpha\,e^{\gamma}\,\phi(d_1)}{S_0\,K\,\sigma}\left(\alpha\,\left(\frac{e^{\gamma}-K}{K}\right) + \alpha\right) \\
&= -\frac{\alpha^2\,e^{2\,\gamma}\,\phi(d_1)}{S_0\,K^2\,\sigma} \\
f''(x) &< 0\;\forall\,x \in (0,1)
\end{aligned}$$

Because $f'(x) > 0$, we know that there exists one and only one $\{p \in (0,1) : f(p) = 0\}$ if and only if

1. $f(0) < 0$
2. $f(1) > 0$

In the following we find the restrictions under which the two points above are achieved for our function f:

1. $f(0) < 0$:

$$\begin{aligned}
f(0) &< 0 \Leftrightarrow \\
K(0)\,e^{-\delta}\,\Phi(-d_2(0)) - S_0(0)\,\Phi(-d_1(0)) &> 0 \Leftrightarrow \\
K(0) &> 0 \Leftrightarrow \\
e^{\gamma} - (1-\alpha)\,e^{\delta} &> 0 \Leftrightarrow \\
\gamma &> \delta + \log(1-\alpha) \qquad (16)
\end{aligned}$$

2. $f(1) > 0$:

$$\begin{aligned}
f(1) &> 0 \Leftrightarrow \\
1 - (K(1)\,e^{-\delta}\,\Phi(-d_2(1)) - 0) &> 0 \Leftrightarrow \\
1 - (e^{\gamma}\,e^{-\delta}\,\Phi(-\infty) - 0) &> 0 \Leftrightarrow \\
1 - e^{\gamma-\delta} &> 0 \Leftrightarrow \\
\gamma &< \delta \qquad (17)
\end{aligned}$$

If $\gamma \in (\delta + \log(1-\alpha), \delta)$, there exists one and only one $\{p \in (0,1) : f(p) = 0\}$. If $\gamma \notin (\delta + \log(1-\alpha), \delta)$, there exists no $\{p \in (0,1) : f(p) = 0\}$.

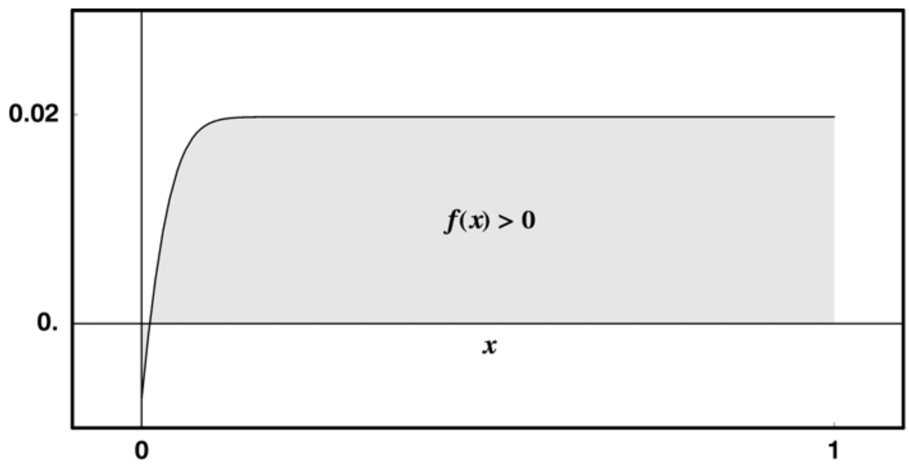

Figure 5: $f(x)$ with the default parameters from the case study.

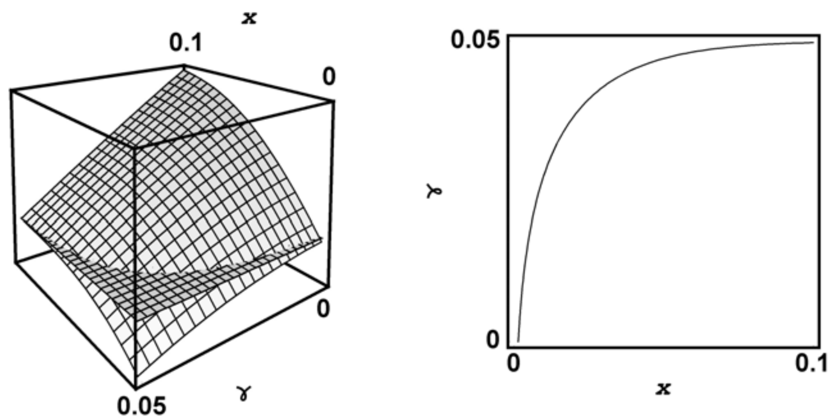

Figure 6: $f(x)$ with the default parameters from the case study exept γ. Left: The surface spanned by f and the $0-$plane. Right: The contour $f(x) = 0$ (the intersection between the surface spanned by f and the $0-$plane.)

If the unequality (16) is not satiesfied, the bond account alone exceeds the minimum level and the guarantee has no potensial value. If the unequality (17) is not satiesfied, the guarantee has a value greater than one, which is not possible, or there are arbitrage possibilities. Therefore we assume that $\gamma \in (\delta + \log(1-\alpha), \delta)$. We can find p by using the numerical method called bisection:

1. Choose $p_{\min} = 0$ as a point assumed to be less than p for all legal parameters and $p_{\max} = 1$ as a point assumed to be greater than p for all legal parameters. Compute
$$p_{test} = \frac{p_{\max} + p_{\min}}{2}$$
2. If $f(p_{test}) > 0$, we set $p_{\max} = p_{test}$, otherwise we set $p_{\min} = p_{test}$.
3. Compute
$$p_{test} = \frac{p_{\max} + p_{\min}}{2}$$
4. Repeat 2 and 3 n times, where n is chosen to achieve a certain accuracy: The width of the interval around p is $(1/2)^n$ and p_{test} in the middle of this interval is an approximation to p.