UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Solution of exam in: STK4500/9500 — Life Insurance and Finance

Day of examination: 10th June 2022

Examination hours: 3:00 pm – 7:00 pm

This problem set consists of 6 pages.

Appendices: None

Permitted aids: Approved calculator

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

This exam consists of four problems. The first one is of theoretical nature, while the three remaining problems are more applied. Make sure to be **precise** and **rigorous** when stating mathematical results and formulae. Please, **write clearly** and avoid crossing out.

Grading: The total score is 16 points which are translated to a number between 0 and 100. The grading scale is F [0,40), E [40,50), D [50, 60), C [60, 70), B [70, 85), A [85,100].

Problem 1 Theory (weight 4 points)

State and prove Thiele's ordinary differential equation for the prospective reserve (when everything is deterministic except for the state of the insured). You may assume for simplicity that the policy functions a_i are a.e. differentiable and continuous. Comment briefly on how you would solve the equation numerically.

Solution: To ease notation write $V_i(t) = V_i^+(t, A)$. Let $\dot{a}_i$ be the a.e. derivative of a_i. Since a_i is continuous $\Delta a_i(t) = 0$ for all t. We can express V_i in a more compact form as

$$V_i(t) = \frac{1}{v(t)} \sum_{j \in \mathcal{S}} \left[\int_t^\infty v(s) p_{ij}(t,s) \theta_j^k(s) ds \right], \tag{1}$$

where θ_j^k is defined as

$$\theta_j^k(s) \triangleq \dot{a}_j(s) + \sum_{\substack{k \in \mathcal{S} \\ k \neq j}} \mu_{jk}(s) a_{jk}(s).$$

Differentiating w.r.t. t and using that $v(t) = e^{-\int_0^t r(s)ds}$ we have

$$V_i'(t) = r(t)V_i(t) + \frac{1}{v(t)} \sum_{j \in \mathcal{S}} \left[\partial_t \int_t^\infty v(s) p_{ij}(t,s) \theta_j^k(s) ds \right].$$

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The derivative of the integral should be treated carefully (chain rule + fundamental theorem of calculus), but we know that

$$\partial_t \int_t^\infty v(s)p_{ij}(t,s)\theta_j^k(s)ds = -v(t)p_{ij}(t,t)\theta_j^k(t) + \int_t^\infty v(s)\partial_t p_{ij}(t,s)\theta_j^k(s)ds.$$

On the one hand $p_{ij}(t,t) = \delta_{i=j}$ and $\partial_t p_{ij}(t,s) = \sum_{k\neq i}\mu_{ik}(t)(p_{ij}(t,s) - p_{kj}(t,s))$ due to Kolmogorov's backward equation.

Until now we have

$$V_i'(t) = r(t)V_i(t) - \theta_i^k(t) + \frac{1}{v(t)}\sum_{j\in\mathcal{S}}\int_t^\infty v(s)\sum_{k\neq i}\mu_{ik}(t)(p_{ij}(t,s) - p_{kj}(t,s))\theta_j^k(s)ds.$$

Pulling out the sum over k and using (1) we get

$$V_i'(t) = r(t)V_i(t) - \theta_i^k(t) + \sum_{k\neq i}\mu_{ik}(t)\left(V_i(t) - V_k(t)\right).$$

Finally, $\theta_i^k(t) = \dot{a}_i(t) + \sum_{\substack{k\in\mathcal{S}\\ k\neq i}}\mu_{ik}(t)a_{ik}(t)$, which implies

$$V_i'(t) = r(t)V_i(t) - \dot{a}_i(t) - \sum_{\substack{k\in\mathcal{S}\\ k\neq i}}\mu_{ik}(t)a_{ik}(t) + \sum_{k\neq i}\mu_{ik}(t)\left(V_i(t) - V_k(t)\right).$$

Merging the sums over $k \neq i$ we finally obtain the equation

$$V_i'(t) = r(t)V_i(t) - \dot{a}_i(t) - \sum_{\substack{k\in\mathcal{S}\\ k\neq i}}\mu_{ik}(t)\left(a_{ik}(t) + V_k(t) - V_i(t)\right).$$

To solve Thiele's equation numerically one could use any of the available numerical methods for solving ODE's such as Euler's method, Taylor's method, Runge-Kutta methods, etc. The simplest is the Euler method which approximates the derivative in time on a grid of points $\{t_k\}_{k=0}^n \subset [0,T]$, $t_0 = 0$, $t_n = T$ such that $\max_{k=1,\ldots,n}\{|t_k - t_{k-1}|\} \to 0$ as $n\to\infty$, usually $t_k = k\Delta t$, $k = 0,\ldots,n$ where $\Delta t = T/n$. Then

$$\partial_t V_i(t) \approx \frac{V_i(t) - V_i(t-\Delta t)}{\Delta t}.$$

Then we use $V_i(T) = \Delta a_i(T)$ to find $V_i(T(n-1)/n)$ and so on. We obtain $V_i(t_k)$ iteratively integrating backwards as,

$$V_i(t_{k+1}) = V_i(t_k) - \Delta t\left[r(t_k)V_i(t_k) - \dot{a}_i(t_k) - \sum_{\substack{j\in\mathcal{S}\\ j\neq i}}\mu_{ij}(t_k)\left(a_{ij}(t_k) + V_j(t_k) - V_i(t_k)\right)\right],$$

for every $k = 0,\ldots,n-1$.

Points: Give 1p for stating the equation correctly. Give 2.5p for the proof and 0.5p for an approach to solve it numerically. The part on the numerical method does not need to be as detailed as above.

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Problem 2 Endowment policy (weight 4 points)

Let $X = \{X_t\}_{t\geq 0}$ be a regular continuous time Markov process with state space $S = \{*, \dagger\}$ where $X_t = *$ means that the insured is active at time t and $X_t = \dagger$ means that the insured is deceased at time t. Denote by $\mu := \mu_{*\dagger} > 0, t \geq 0$ the intensity rate of transitioning from state $*$ to state $\dagger$ which, for simplicity, we take to be constant (NB! this is far from realistic, but this is a written exam without computers).

Consider an endowment insurance contract for a $x_0 \geq 0$ years old person with expiry at age $x_0 + T$, $T \geq 0$ being the contract length. This contract pays the benefit B in case that the insured survives by the end of the contract and a death benefit C in case the insured dies during the period of the contract. Assume constant interest rate r.

(a) Show that the value of this insurance at any time $t \in [0, T]$ is given by

$$V_*^+(t, A) = Be^{-(r+\mu)(T-t)} + C\frac{\mu}{r+\mu}\left(1 - e^{-(r+\mu)(T-t)}\right), \quad t \in [t_0, T].$$

Solution: The policy functions determining entirely this insurance are

$$a_*(t) = \begin{cases} B \text{ if } t \geq T \\ 0 \text{ else} \end{cases} \qquad a_{*\dagger}(t) = \begin{cases} C \text{ if } t \in [0, T) \\ 0 \text{ else.} \end{cases}$$

The value of this insurance is given by

$$\begin{aligned} V_*^+(t, A) &= V_*(t, A_*) + V_*(t, A_{*\dagger}) \\ &= \frac{1}{v(t)} \int_t^\infty v(s) p_{**}(x_0 + t, x_0 + s) da_*(s) \\ &\quad + \frac{1}{v(t)} \int_t^\infty v(s) p_{**}(x_0 + t, x_0 + s) \mu_{*\dagger}(x_0 + s) a_{*\dagger}(s) ds \\ &= Be^{-r(T-t)} p_{**}(x_0 + t, x_0 + T) + Ce^{rt} \int_t^T e^{-rs} e^{-\mu(s-t)} \mu ds \\ &= Be^{-(r+\mu)(T-t)} + C\frac{\mu}{r+\mu}\left(1 - e^{-(r+\mu)(T-t)}\right), \quad t \in [0, T]. \end{aligned}$$

Points: Give 1p for writing down the general formula and 1p for setting in the variables and simplifying. Give 0.25p for the policy functions only. Subtract 0.25p per computational error.

(b) Compute the (fair) single premium at the beginning of the contract and the (fair) yearly premiums.

Solution: The single premium is the value of the insurance at the beginning of the contract. We denote by π_0 the single premium for a x_0 years old person. Therefore

$$\pi_0 \triangleq V_*^+(0, A) = Be^{-(r+\mu)T} + C\frac{\mu}{r+\mu}\left(1 - e^{-(r+\mu)T}\right).$$

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We introduce a yearly payment of premiums π in this policy. Therefore, the policy functions now look like

$$a_*(t) = \begin{cases} -\pi t \text{ if } t \in [0, T) \\ -\pi T + B \text{ if } t \geq T \end{cases} , \quad a_{*\dagger}(t) = \begin{cases} C \text{ if } t \in [0, T) \\ 0 \text{ else.} \end{cases}$$

In this case, we have

$$\begin{aligned} V_*^+(t, A) = & \ V_*(t, A_*) + V_*(t, A_{*\dagger}) \\ = & \frac{1}{v(t)} \int_t^\infty v(s) p_{**}(t, s) da_*(s) + \frac{1}{v(t)} \int_t^\infty v(s) p_{**}(t, s) \mu_{*\dagger}(s) a_{*\dagger}(s) ds \\ = & -\pi e^{rt} \int_t^T e^{-rs} p_{**}(t, s) ds + B e^{-(r+\mu)(T-t)} \\ & + C \frac{\mu}{r+\mu} \left(1 - e^{-(r+\mu)(T-t)}\right) \\ = & -\pi \frac{1}{r+\mu} \left(1 - e^{-(r+\mu)(T-t)}\right) + B e^{-(r+\mu)(T-t)} \\ & + C \frac{\mu}{r+\mu} \left(1 - e^{-(r+\mu)(T-t)}\right). \end{aligned}$$

where we used item (b) and added the premiums.

Using the equivalence principle we must impose that $V_*^+(0, A) = 0$. Then it turns out that

$$\pi = \frac{B e^{-(r+\mu)T} + C \frac{\mu}{r+\mu} \left(1 - e^{-(r+\mu)T}\right)}{\frac{1}{r+\mu} \left(1 - e^{-(r+\mu)T}\right)}.$$

Points: Give 0.5p for the single premium and 1p for writing down the present value with premiums. Give 0.5p for finding π in closed form. Give 0.25p for the policy functions only. Subtract 0.25p per computational error.

Problem 3 Surrender policy (weight 4 points)

A healthy life of age x_0 writes an endowment insurance with contract length $T > 0$. The death benefit corresponds to the current prospective reserve at the time of death and the endowment benefit is a fixed lump sum of E units. The policyholder pays a single π_0 at the beginning of the contract.

We allow the policyholder to surrender the policy at any time. In such case, we pay them a benefit corresponding to the current prospective reserve at the surrendering time with a penalty, i.e. $(1 - \varepsilon)$, $\varepsilon \in [0, 1]$, multiplied by the prospective reserve at the surrendering time.

Let $\mathcal{S} = \{*, \dagger, s\}$ be the states of this insurance being $*$ alive, $\dagger$ deceased and s surrendered. Let $\mu_{*\dagger}$ and μ_{*s} denote the transition rates to *deceased* and *surrendered*, respectively, hereby assumed to be constant.

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Prove that the prospective reserve for this policy coincides with the prospective reserve of a pure endowment policy with interest rate r and with respect to a Markov chain X^ε with state space $\mathcal{S}^\varepsilon = \{*, \dagger^\varepsilon\}$ with $\mu_{*\dagger^\varepsilon} = \varepsilon\mu_{*s}$. Could you give an interpretation of why?

Solution: The policy functions for this insurance are $a_{*\dagger}(t) = V_*(t)$ and $a_{*s}(t) = (1-\varepsilon)V_*(t)$, where $V_*(t)$ denotes the prospective reserve at time t. Thiele's ODE becomes

$$V_*'(t) = rV_*(t) - \mu_{*\dagger}(a_{*\dagger}(t) - V_*(t)) - \mu_{*s}(a_{*s}(t) - V_*(t)), \tag{2}$$

with final condition $V_*(T) = E$. Observe that the middle term on the right-hand side cancels out and $a_{*s}(t) = (1-\varepsilon)V_*(t)$ giving rise to

$$V_*'(t) = (r + \varepsilon\mu_{*s})\, V_*(t).$$

The above is a homogeneous linear equation which can be easily solved. With $V_*(T) = E$ we obtain

$$V_*(t) = Ee^{-(r+\varepsilon\mu_{*s})(T-t)}, \quad t \in [0, T]. \tag{3}$$

Observe that in our setting, the term $e^{-\varepsilon\mu_{*s}(T-t)}$ corresponds to the transition probability $p_{*\dagger^\varepsilon}(t, T)$, hence we can express $V_*(t)$ as

$$V_*(t) = E\frac{v(T)}{v(t)}p_{*\dagger^\varepsilon}(t, T),$$

where $v(t) = e^{-rt}$.

The intuition is that we do not need to reserve anything for transitions to $\dagger$ in (2) since the payout precisely corresponds to the accumulated (fair) reserve. On the other hand, we accumulate a slight amount of reserve, i.e. $\varepsilon\mu_{*s}V_*(t)$, since the payout is always below the current reserve due to penalization, which produces a positive sign in (2). Hence, the reserve can simply be seen as a pure endowment, $V_*(T) = E$ where a proportion of surrenders $\varepsilon\mu_{*s}$ will drop out without endowment, thus needing less reserve for the survivors. The formula in (3) can also be seen as a savings account for saving the amount E where the return is slightly higher than r due to the possibility of surrenders not having E.

Points: Give 1p for writing down the policy functions and the present value for the original three-state policy. Give 2p for writing down Thiele's equation and solving it. Give the final point for identifying the equivalence with the two-state policy and the interpretation. Subtract 0.25p per computational error and subtract 0.5p per theoretical error. A solution without Thiele's equation is equally valid.

Problem 4 Unit-linked policy (weight 4 points)

Consider a (continuous) permanent disability model with (constant) $\mu_{*\diamond} = 0.0279$, $\mu_{*\dagger} = 0.0229$ and $\mu_{\diamond\dagger} = \mu_{*\dagger} = 0.0229$ for a life of age $x_0 = 30$. The

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term of the policy is $T = 10$ years. The disability pension is financed by the value of a fund described by a stochastic process $S = \{S_t, t \in [0, T]\}$.

Assuming that the initial investment is $S_0 = 100\,000$ and that interest rate is $r = 3\%$, compute the exact numerical value of the single premium π_0 of this insurance. Hint: solving Kolmogorov's equations with constant rates one can arrive at $p_{**}(t,s) = e^{-(\mu_{*\diamond}+\mu_{*\dagger})(s-t)}$, $p_{\diamond\diamond}(t,s) = e^{-\mu_{\diamond\dagger}(s-t)}$ and $p_{*\diamond}(t,s) = \frac{\mu_{*\diamond}}{\mu_{*\dagger}} e^{-\mu_{*\diamond}(s-t)} \left[1 - e^{-\mu_{*\dagger}(s-t)}\right]$.

Solution: There are three states $\mathcal{S} = \{*, \diamond, \dagger\}$. This is a unit-linked policy for a (permanent) disability where the disability benefit is funded by the value of an underlying fund S. A single premium of π_0 is paid at the beginning of the contract. The (non-zero) policy function of this UL insurance is: $g_\diamond(t, S_t) = S_t$.

The mathematical reserves are then given by

$$V_*^+(t,A) = \frac{1}{v(t)} \int_t^T v(s) p_{*\diamond}(x_0+t, x_0+s) \mathbb{E}_{\mathbb{Q}}[S_s|\mathcal{F}_t] ds.$$

$$V_\diamond^+(t,A) = \frac{1}{v(t)} \int_t^T v(s) p_{\diamond\diamond}(x_0+t, x_0+s) \mathbb{E}_{\mathbb{Q}}[S_s|\mathcal{F}_t] ds.$$

In the equations above, $\mathbb{Q}$ is an EMM (or risk-neutral measure) for which discounted S is a $\mathbb{Q}$ martingale, i.e. the process $t \mapsto v(t)S_t$ is a $\mathbb{Q}$-martingale implying that for every $s \geq t$ we have $\mathbb{E}_{\mathbb{Q}}[v(s)S_s|\mathcal{F}_t] = v(t)S_t$. As a result we have

$$V_*^+(t,A) = S_t \int_t^T p_{*\diamond}(x_0+t, x_0+s) ds.$$

$$V_\diamond^+(t,A) = S_t \int_t^T p_{\diamond\diamond}(x_0+t, x_0+s) ds.$$

The single premium π_0 is the initial value of the insurance assuming that the insured enters the contract in state $*$, i.e. $V_*^+(0,A)$. Hence,

$$\pi_0 = V_*^+(0,A) = S_0 \int_0^T p_{*\diamond}(x_0, x_0+s) ds.$$

Now,

$$\begin{aligned}
\pi_0 &= S_0 \int_0^T p_{*\diamond}(x_0, x_0+s) ds = S_0 \frac{\mu_{*\diamond}}{\mu_{*\dagger}} \int_0^T e^{-\mu_{*\diamond}s}(1 - e^{-\mu_{*\dagger}s}) ds \\
&= S_0 \frac{\mu_{*\diamond}}{\mu_{*\dagger}} \left[\frac{1}{\mu_{*\diamond}} \left(1 - e^{-\mu_{*\diamond}T}\right) - \frac{1}{\mu_{*\diamond}+\mu_{*\dagger}} \left(1 - e^{-(\mu_{*\diamond}+\mu_{*\dagger})T}\right) \right] \\
&= 100\,000 \frac{0.0279}{0.0229} \left[\frac{1}{0.0279} \left(1 - e^{-0.279}\right) - \frac{1}{0.0508} \left(1 - e^{-0.508}\right) \right] \\
&\approx 107\,893\,.
\end{aligned}$$

Points: Give 1p for writing down the present value V_*^+ in general. Give 2p for additionally arguing that $\mathbb{E}_{\mathbb{Q}}[v(s)S_s|\mathcal{F}_t] = v(t)S_t$. Give 1p for mentioning that $\pi_0 = V_*^+(0,A)$ and computing the final numerical value with steps. Subtract 0.25p per computational error.

GOOD LUCK!