Appendix

Not necessarily all formulas are needed to solve the problems in the exam.

1. Forward Kolmogorov equation:

$$\frac{d}{dt}p_{ij}(s,t) = -p_{ij}(s,t)\,\mu_j(t) + \sum_{k\neq j} p_{ik}(s,t)\,\mu_{kj}(t), \quad i,j \in S,$$

with $p_{ij}(s,s) = 0$, if $i \neq j, p_{ij}(s,s) = 1$, if $i = j$, where $\mu_j(t) = \sum_{k\neq j}\mu_{jk}(t)$.

2.

$$p_{jj}(s,t) = \bar{p}_{jj}(s,t) = \exp\left(-\sum_{k\neq j}\int_s^t \mu_{jk}(u)\,du\right), \quad j \in S.$$

3. Thiele's difference equation:

$$V_i^+(t) = a_i^{\text{Pre}}(t) + \sum_{j\in S} e^{-\delta} p_{ij}(t,t+1)\left\{a_{ij}^{\text{Post}}(t) + V_j^+(t+1)\right\},$$

where $a_i^{\text{Pre}}(t)$ (pension payments) and $a_{ij}^{\text{Post}}(t)$ (benefit payments) are the policy functions.

4. Prospective reserve (in continuous time):

$$V_j^+(t) = \frac{1}{v(t)}\sum_{g\in S}\int_{(t,\infty)} v(s)\,p_{jg}(t,s)\left\{da_g(s) + \left(\sum_{h\in S, h\neq g} a_{gh}(s)\,\mu_{gh}(s)\right)ds\right\},$$

for $j \in S$.

5. Black-Scholes model:

$$S_t = S_0 + \int_0^t \mu S_u du + \int_0^t \sigma S_u dB_u, \quad 0 \le t \le T,$$

where $S_0 > 0, \sigma \neq 0$ and μ are constants and $B_t, 0 \le t \le T$ is a Brownian motion.

6. Pricing formula for a T-claim X :

$$\begin{aligned}\text{ClaimValue}_t &= \mathbb{E}_{\tilde{P}}\left[e^{-r(T-t)}X\middle|\,\mathcal{G}_t\right], \qquad 0 \le t \le T,\\ \text{ClaimValue}_0 &= \mathbb{E}_{\tilde{P}}\left[e^{-rT}X\right],\end{aligned}$$

where $\tilde{P}$ (equivalent martingale measure or risk neutral probability measure) is the probability measure such that $\tilde{S}_t = e^{-rt}S_t, 0 \le t \le T$ is a martingale under $\tilde{P}$ and $\mathcal{G}_t = \sigma(S_u, 0 \le u \le t)$.

7. Vasicek model:

$$r(t) = r(0) + \int_0^t \alpha(\beta - r(u))\, du + \sigma B_t,$$

for non-negative constants $r(0), \alpha, \beta$ and σ.

8. Zero-coupon bond value at time $t = 0$ (in the Vasicek model):

$$P(0,T) = \exp(-T \cdot R(T, r(0))),$$

where

$$R(s,x) = \beta - \frac{\lambda\sigma}{\alpha} + \frac{\sigma^2}{2\alpha^2} - \frac{1}{\alpha s}\left\{\left(\beta - \frac{\lambda\sigma}{\alpha} + \frac{\sigma^2}{2\alpha^2} - x\right)\left(1 - e^{-\alpha s}\right) - \frac{\sigma^2}{4\alpha^2}\left(1 - e^{-\alpha s}\right)^2\right\},$$

and λ is the risk premium.

9. Integration by parts formula: Let

$$X_t = X_0 + \int_0^t K_u du + \int_0^t H_u dB_u, \quad Y_t = Y_0 + \int_0^t \tilde{K}_u du + \int_0^t \tilde{H}_u dB_u,$$

where $B_u, 0 \leq u \leq t$ is a Brownian motion. Then

$$X_t Y_t = X_0 Y_0 + \int_0^t X_u dY_u + \int_0^t Y_u dX_u + [X,Y]_t,$$

where

$$[X,Y]_t = \int_0^t H_u \tilde{H}_u du.$$

10. Itô's formula: Let

$$X_t = X_0 + \int_0^t K_u du + \int_0^t H_u dB_u,$$

where $B_u, 0 \leq u \leq t$ is a Brownian motion and $f \in C^2(\mathbb{R})$. Then

$$f(X_t) = f(X_0) + \int_0^t \frac{d}{dx} f(X_u)\, dX_u + \frac{1}{2}\int_0^t \frac{d^2}{dx^2} f(X_u)\, H_u^2 du.$$

11. Moment generating function of $X \sim \mathcal{N}(\mu, \sigma^2)$:

$$\mathbb{E}\left[e^{\theta X}\right] = \exp\left(\mu\theta + \frac{\sigma^2\theta^2}{2}\right).$$