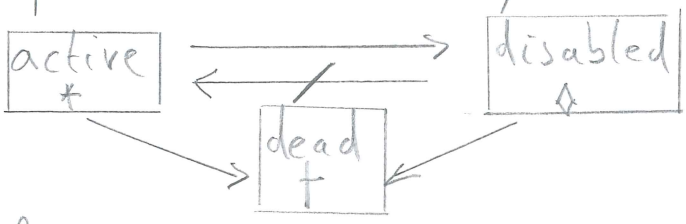
STK 4500

Prøveeksamen

Prob. 1: permanent disability insurance:



$V_*^+(37, A_\diamond^{Pre})$?

→ solution to this problem consists of two steps:

1. step: calculation of $P_{ij}(s,t)$:

→ idea: forward Kolmogorov equation (Th. 2.2.2.3)

$$\frac{d}{dt} P_{ij}(s,t) = -P_{ij}(s,t)\mu_j(t) + \sum_{k \neq j} P_{ik}(s,t)\mu_{kj}(t)$$

with

$$\mu_j(t) = \sum_{k \neq j} \mu_{jk}(t)$$

$i = *, j = \diamond$ ⟹

$$\frac{d}{dt} P_{*\diamond}(s,t) = -P_{*\diamond}(s,t)\underbrace{\mu_\diamond(t)}_{=\mu_{\diamond *}(t)+\mu_{\diamond \dagger}(t)} + P_{**}(s,t)\mu_{*\diamond}(t) + P_{*\dagger}(s,t)\underbrace{\mu_{\dagger\diamond}(t)}_{=0} \quad (+)$$

⟹

$$\frac{d}{dt} P_{*\diamond}(s,t) + P_{*\diamond}(s,t)\cdot\mu_{\diamond\dagger}(t) = P_{**}(s,t)\mu_{*\diamond}(t)$$

$\cdot \exp\left(\int_s^t \mu_{\diamond\dagger}(u)du\right)$ on both sides

$\frac{d}{dt}\int_s^t \mu_{\diamond\dagger}(u)du = \mu_{\diamond\dagger}(t)$

$$\left(\frac{d}{dt} P_{*\diamond}(s,t) + P_{*\diamond}(s,t)\cdot\mu_{\diamond\dagger}(t)\right)\exp\left(\int_s^t \mu_{\diamond\dagger}(u)du\right)$$

Product rule

$$\frac{d}{dt}\left(P_{*\diamond}(s,t)\cdot\exp\left(\int_s^t \mu_{\diamond\dagger}(u)du\right)\right)$$

$\int_s^T (\cdot)\,dt$ ⟹

$$P_{**}(s,t)\mu_{*\diamond}(t)\cdot\exp\left(\int_s^t \mu_{\diamond\dagger}(u)du\right)$$

⟹ on both sides

$$P_{*\diamond}(s,T)\exp\left(\int_s^T \mu_{\diamond\dagger}(u)du\right) = \int_s^T P_{**}(s,t)\mu_{*\diamond}(t)\exp\left(\int_s^t \mu_{\diamond\dagger}(u)du\right)dt$$

⟹ dividing by $\exp(\cdot)$ on both sides

$$P_{*\diamond}(s,T) = \int_s^T \underbrace{P_{**}(s,t)}_{\text{Th. 2.2.2.4}}\mu_{*\diamond}(t)\exp\left(-\int_t^T \mu_{\diamond\dagger}(u)du\right)dt \quad (*)$$

$$= \exp\left(-\int_s^t (\mu_{*\dagger}(u)+\mu_{*\diamond}(u))du\right)$$

$$= \exp(-(t-s)\cdot 0.0021)$$

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(2)

$$\overset{(*)}{\Longrightarrow} \quad p_{*\diamond}(s,T) = 0.0006 \int_s^T \exp(-(t-s)\cdot 0.0021)\cdot \exp(-(T-t)\cdot 0.0015)\,dt$$

$$= \exp(-(T-s)\cdot 0.0015)\cdot(1-\exp(-(T-s)\cdot 0.0006))$$

Further:

$$p_{*\dagger}(s,t) = 1 - p_{**}(s,t) - p_{*\diamond}(s,t), \quad p_{\diamond\diamond}(s,t) \overset{\text{Th. 2.2.2.4}}{=} \exp\left(-\int_s^t \mu_{\diamond\dagger}(u)\,du\right) =$$

$$= \exp(-(t-s)\cdot 0.0015), \quad p_{\diamond\dagger}(s,t) = 1 - p_{\diamond\diamond}(s,t) - \underbrace{p_{\diamond *}(s,t)}_{=0}$$

(alternatively one could guess the solutions to (†))

2. step: Application of Thiele's difference equation:

Policy function:

$$a_{\diamond}^{Pre}(t) = \begin{cases} 12000, & \text{if } 20 \le t < 40 \\ 0 & \text{else} \end{cases} \leftarrow \text{disability pension payments}$$

Thiele's diff. equation

$$V_*^{\dagger}(t) = \underbrace{a_*^{Pre}(t)}_{=0} + e^{-\delta}\underbrace{p_{**}(t,t+1)}_{e^{-0.0015}}\cdot\left\{\underbrace{a_{**}^{Post}(t)}_{=0} + V_*^{\dagger}(t+1)\right\}$$

$$+ e^{-\delta} p_{*\dagger}(t,t+1)\cdot\left\{\underbrace{a_{*\dagger}^{Post}(t)}_{=0} + \underbrace{V_{\dagger}^{\dagger}(t+1)}_{=0}\right\}$$

$$+ e^{-\delta} p_{*\diamond}(t,t+1)\cdot\left\{\underbrace{a_{*\diamond}^{Post}(t)}_{=0} + V_{\diamond}^{\dagger}(t+1)\right\}$$

$$= e^{-\delta} p_{**}(t,t+1)\, V_*^{\dagger}(t+1) + e^{-\delta} p_{*\diamond}(t,t+1)\cdot V_{\diamond}^{\dagger}(t+1) \qquad (**)$$

and

$$V_{\diamond}^{\dagger}(t) = a_{\diamond}^{Pre}(t) + e^{-\delta} p_{\diamond\diamond}(t,t+1)\, V_{\diamond}^{\dagger}(t+1) \quad \text{(similarly)}$$

with terminal values: $V_*^{\dagger}(40) = V_{\diamond}^{\dagger}(40) = 0$

$$\longrightarrow \quad V_{\diamond}^{\dagger}(39) = 12000, \quad V_{\diamond}^{\dagger}(38) = 23627.9 \qquad V_*^{\dagger}(37, A_{\diamond}^{Pre})$$

$$\overset{(**)}{\Longrightarrow} \quad V_*^{\dagger}(39) = 0, \quad V_*^{\dagger}(38) = 6.97464, \quad \underline{V_*^{\dagger}(37) = 20.4873\ \$}$$

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Prøveeksamen

Prob. 2: 10-year pure endowment with stoch. int. rates $r(t)$

$\longrightarrow$ **policy function** (in continuous time)

$$a_*(t) = \begin{cases} 115000, & \text{if } t = 65 \\ 0 & \text{else} \end{cases} \leftarrow \text{endowment payment}$$

Recall the classical case (constant int. rates δ)

(8.1.20) for $\pi = 0$ (premium) $\rightarrow$ $V_*^+(0) = \underbrace{115000 \cdot e^{-\delta \cdot T}}_{\text{def. } PV_0} {}_{T}p_{x,x}$ ($x = 55$ years, $T = 10$ years)

$PV_0 \leftarrow$ present value at $t=0$ of the endowment payment paid at maturity T

However, in the case of stoch. int. rates we have to replace PV_0 by the present value $\pi_0^*(T)$ at time $t=0$ of $a_*(T)$ based on the stoch. $r(t)$

Pricing formula for bonds (8.1.21), $T = 10$:

$$\pi_0^*(T) = \text{Claim Value}_0 = E_{\tilde{P}}\left[\exp\left(-\int_0^T r(u)\,du\right) \cdot \underbrace{a_*(T)}_{=115000} \,\middle|\, \mathcal{G}_0\right]$$

$$= \underbrace{E_{\hat{P}}\left[\exp\left(-\int_0^T r(u)\,du\right)\right]}_{\text{Exerc. II, Prob. 1: } P(0,T) \leftarrow \text{price of a zero-coupon bond at } t=0} \cdot 115000$$

$$\Rightarrow \quad V_*^+(0) = 115000 \cdot P(0,T) \cdot \underbrace{{}_{T}p_{x,x}}_{= \exp\left(-\int_x^{x+T} \mu_{x+t}(u)\,du\right) = \exp(-10 \cdot 0.015)}$$

where $P(0,T) \overset{\text{Hint}}{=} \underbrace{\exp(-T \cdot R(T, r(0)))}_{= 0.557561}$

$$\Rightarrow \quad V_*^+(0) = 55188.1 \text{ \$}$$

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Prob. 3 : 10-year unit-linked term insurance

→ **policy function** :

$$a_{*+}(t) = \begin{cases} C(t-45), & \text{if } [45,55) \\ 0 & \text{else} \end{cases} \quad \leftarrow \text{benefit payment}$$

where

$$C(t) = \underbrace{(1-0.045)}_{\text{portion of } P \text{ invested in the fund}} \cdot \overbrace{1.2}^{120\%} \cdot P \cdot \overbrace{S_t}^{\text{fund value at time } t} \cdot \underbrace{e^{-0.007 \cdot t}}_{\substack{\text{discount factor} \\ \text{for the management} \\ \text{charges}}} \qquad (1)$$

$P = 10000\,\$$

explicit formula for prospective reserves (see also Exercise 9, Prob. 5)

$$V_*^+(0) = \int_0^{10} \underbrace{\pi_0^{*+}(s)}_{e^{-s\cdot\delta}\cdot a_{*+}(s) \text{ in the classical case}} p_{**}(45, 45+s) \cdot \mu_{*+}(45+s)\, ds, \qquad (2)$$

where

$$\pi_0^{*+}(s) = (\text{ClaimValue}_0 \overset{\text{pricing formula}}{=} E_{\hat{P}}[e^{-r\cdot s} C(s)], \qquad (3)$$

where $\hat{P}$ is the prob. meas. such that $\tilde{S}_t \overset{\text{def}}{=} e^{-rt} S_t$ is a martingale under $\hat{P}$, that is

$$E_{\hat{P}}[\tilde{S}_s \mid \mathcal{G}_t] = \tilde{S}_t, \quad s \geq t$$

$$\Rightarrow \underbrace{E_{\hat{P}}[\tilde{S}_s \mid \mathcal{G}_0]}_{= E_{\hat{P}}[\tilde{S}_s]} = \tilde{S}_0 = 1 \qquad (4)$$

(1), (3), (4) ⇒

$$\pi_0^{*+}(s) = (1-0.045)\cdot 1.2 \cdot \overbrace{10000}^{=P} \cdot e^{-0.007\cdot s} \underbrace{E_{\hat{P}}[e^{-r\cdot s}\overbrace{\tilde{S}_s}^{=\hat{S}_s}]}_{=\tilde{S}_0 = 1}$$

(2) ⇒

$$V_*^+(0) = (1-0.045)\cdot 1.2 \cdot 10000 \int_0^{10} e^{-0.007\cdot s} \cdot \underbrace{p_{**}(45, 45+s)}_{\exp(-\int_{45}^{45+s} \mu_{*+}(u)\,du) = \exp(-0.0065\cdot s)} \cdot \overbrace{0.0065}^{\mu_{*+}(45+s)}\, ds$$

$$= (1-0.045)\cdot 1.2 \cdot 10000 \cdot 0.0065 \cdot \left(\frac{-1}{0.0135} e^{-0.0135\cdot s} \Big|_{s=0}^{10} \right)$$

$$= 696.81\,\$$$

(5)

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Prob. 4 : policy functions :

$$a_*(t) = \begin{cases} -P\cdot(t-50), & \text{if } t\in[50,52) \\ -P\cdot(52-50), & \text{if } t\geq 52 \end{cases}$$ ← premium payments

$$a_\downarrow(t) = \begin{cases} 60000\cdot(t-50), & \text{if } t\in[50,52) \\ 60000\cdot(52-50), & \text{if } t\geq 52 \end{cases}$$ ← disability pension payment

Th. 4.19 $\Rightarrow$

$$V_*^+(0) = \int_{(50,\infty)} e^{-(s-50)\cdot\delta}\, p_{**}(50,s)\, da_*(s) = \dot{a}_*(s)\,ds = (-P)\cdot 1_{[50,52)}(s)$$ ← derivative

$$+ \int_{(50,\infty)} e^{-(s-50)\cdot\delta}\, p_{*\downarrow}(50,s)\, da_\downarrow(s) = \dot{a}_\downarrow(s)\,ds = 60000\cdot 1_{[50,52)}(s)$$

$$\overset{\text{substitution}}{=} \int_0^2 e^{-s\cdot\delta}\, p_{**}(50,50+s)\,(-P)\,ds$$ ← known

$$+ \int_0^2 e^{-s\cdot\delta}\, p_{*\downarrow}(50,50+s)\cdot 60000\,ds$$

$$= 1 - p_{**}(50,50+s) - p_{*\dagger}(50,50+s)$$

$$= -\frac{2}{3}\exp(-0.015\cdot s) + \frac{2}{3}\exp(-0.01\cdot s)$$

$V_*^+(0) = 0$ equivalence principle

$$\Rightarrow P = \frac{60000\int_0^2 e^{-s\cdot\delta}\, p_{*\downarrow}(50,50+s)\,ds}{\int_0^2 e^{-s\cdot\delta}\, p_{**}(50,50+s)\,ds} = 195.99\ \$$$