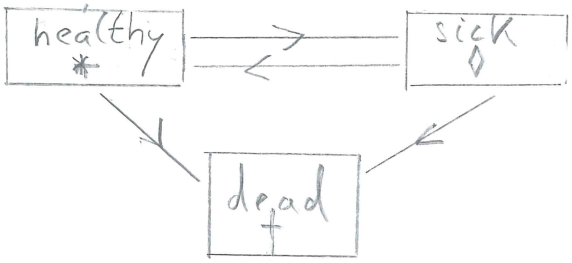
STK 4500

Exercises 2

Prob. 1:



$P_{**}(x, x+10)$, $P_{*\diamond}(x, x+10)$? for $x = 60$ years

idea: discretized version of the forward Kolmogorov equation:

$$\frac{d}{dt} P_{ij}(s,t) = -P_{ij}(s,t)\mu_j(t) + \sum_{k \neq j} P_{ik}(s,t)\mu_{kj}(t) \qquad (*)$$

$$\frac{d}{dt} P_{ij}(s,t) \approx \frac{P_{ij}(s,t+h) - P_{ij}(s,t)}{h}, \text{ if } h > 0 \text{ is small}$$

$\longrightarrow$ Euler-approximation of $(*)$: recursion:

$$P_{ij}(s, s+t-s+h) = P_{ij}(s, s+t-s) + h\left(-P_{ij}(s,s+t-s)\mu_j(s+t-s) + \sum_{k \neq j} P_{ik}(s,s+t-s)\mu_{kj}(s+t-s)\right) \qquad (**)$$

for $m \overset{\text{def}}{=} t-s = 0, h, 2h, \ldots, T-h$

$$\mu_i(t) = \sum_{k \neq i} \mu_{ik}(t)$$

$s = 60$, $T = 10$:

$$P_{**}(60, 60+m+h) = P_{**}(60, 60+m) - h\, P_{**}(60, 60+m)\cdot(\mu_{*\diamond}(60+m) + \mu_{*\dagger}(60+m)) + h\, P_{*\diamond}(60, 60+m)\mu_{\diamond *}(60+m) \qquad (1)$$

and

$$P_{*\diamond}(60, 60+m+h) = P_{*\diamond}(60, 60+m) - h\, P_{*\diamond}(60, 60+m)(\mu_{\diamond\dagger}(60+m) + \mu_{\diamond *}(60+m)) + h\, P_{**}(60, 60+m)\mu_{*\diamond}(60+m) \qquad (2)$$

for $m = 0, h, 2h, \ldots, 10-h$ years with initial values $P_{**}(60,60) = 1$ and $P_{*\diamond}(60,60) = 0$

Choose the step size $h = \frac{1}{12}$ (1 month).

$\mu_{*\diamond}(60) = 0.01420$, $\mu_{*\dagger}(60) = 0.01495$, $\mu_{\diamond *}(60) = 0.00142$, $\mu_{\diamond\dagger}(60) = 0.01495$

$(1),(2) \longrightarrow$ $P_{**}(60, 60+\frac{1}{12}) = 0.99757$, $P_{*\diamond}(60, 60+\frac{1}{12}) = 0.00118$

$(1),(2) \longrightarrow \cdots$ $P_{**}(60, 60+10) = 0.58756$, $P_{*\diamond}(60, 60+10) = 0.20263$

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Exercises 2

Prob. 2: $R(t,t+0.25), \dots, R(t,t+2)$?

Recall:

$$B(t) = \sum_{i=1}^{m} c_i \cdot e^{-(T_i - t)\cdot R(t,T_i)} + X \cdot e^{-(T_m - t)\cdot R(t,T_m)}$$

(c_i: coupon payment at T_i; X: principal)

$\longrightarrow$ PV of a coupon bond

1. bond: $B_1(t) = 98.7 = 100 \cdot e^{-0.25 \cdot R(t,t+0.25)}$

$\Rightarrow R(t,t+0.25) = 0.05234$

2. bond: $B_2(t) = 97.3 = 100 \cdot e^{-0.5 \cdot R(t,t+0.5)}$

$\Rightarrow R(t,t+0.5) = 0.05474$

3. bond: $B_3(t) = 94.6 = 100 \cdot e^{-1 \cdot R(t,t+1)}$

$\Rightarrow R(t,t+1) = 0.0555$

4. bond: $B_4(t) = 102.4 =$

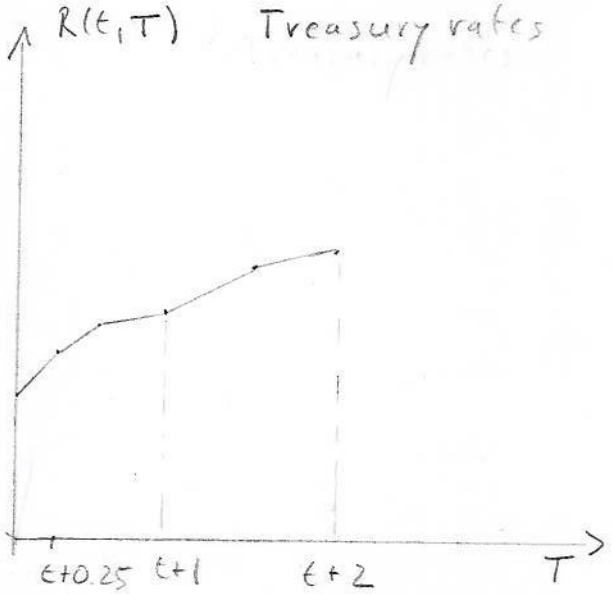
$$\frac{8}{2} \cdot e^{-0.5 \cdot R(t,t+0.5)} + \frac{8}{2} e^{-1 \cdot R(t,t+1)} + \left(\frac{8}{2} + 100\right) e^{-1.5 \cdot R(t,t+1.5)}$$

$\Rightarrow R(t,t+1.5) = 0.06228$

5. bond: $B_5(t) = 110.1 =$

$$\frac{12}{2} e^{-0.5 R(t,t+0.5)} + \frac{12}{2} e^{-1 \cdot R(t,t+1)} + \frac{12}{2} e^{-1.5 R(t,t+1.5)} + \left(\frac{12}{2} + 100\right) e^{-2 \cdot R(t,t+2)}$$

$\Rightarrow R(t,t+2) = 0.06476$



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Exercices 2

Prob. 3: endowment

→ policy functions (in continuous time):

$$a_*(t) = \begin{cases} 0, & \text{if } 0 \le t < 25 \\ -2200 \cdot (t-25), & \text{if } t \in [25, 65) \\ (+2200)(65-25) + 125000, & \text{if } t \ge 65 \end{cases}$$

(125000 ↑ endowment amount)

$$a_{*\dagger}(t) = \begin{cases} 250000, & \text{if } t \in [25, 65) \\ 0 \text{ else} \end{cases}$$

Def. 4.11 →

$$V^{\dagger}(30, A) = \frac{1}{V(30)} \int_{(30,\infty)} V(s)\, d\big(A_*(s,\omega)\big) \overset{(5.26)}{=} \int_0^s \mathbb{1}_{\{X_u(\omega)=*\}}\, da_*(u)$$

$$+ \frac{1}{V(30)} \int_{(30,\infty)} V(s)\, d\big(A_{*\dagger}(s,\omega)\big) \overset{(5.27)}{=} \int_0^s a_{*\dagger}(u)\, dN_{*\dagger}(u,\omega)$$

$$= \frac{1}{V(30)} \left(\int_{(30,\infty)} V(s) \cdot \mathbb{1}_{\{X_s(\omega)=*\}}\, da_*(u) \right) + \frac{1}{V(30)} \left(\int_{(30,\infty)} V(s)\, a_{*\dagger}(s)\, dN_{*\dagger}(s,\omega) \right)$$

$$= \int_{(30,45)} V(s)\, \underbrace{da_*(u)}_{= \dot{a}_*(u)\, du} \qquad = \sum_{30 \le s < \infty} V(s)\, a_{*\dagger}(s)\, \mathbb{1}_{\{X_{s-}(\omega)=*,\, X_s(\omega)=\dagger\}}$$

$$= V(45)\, a_{*\dagger}(45)$$

$$= e^{30 \cdot \delta} \int_{30}^{45} e^{-s \cdot \delta} (-2200)\, ds + e^{-15 \cdot \delta} \underbrace{a_{*\dagger}(45)}_{= 250000}$$

$$= 133810 \text{ \$}$$