

11. Policies with stochastic interest rates

11.1

Goal: Calculation of prospective reserves $V_{\overline{T}}^+(t, A)$ with respect to stochastic technical interest rates.

We will assume that $a_j(t) = \int_0^t \dot{a}_j(s) ds$

We know that the classical prospective reserve is given by

$$V_{\overline{T}}^+(t, A) = V_j^+(t) \big|_{j=X_t} \quad (11.1)$$

with

$$V_j^+(t) = \sum_{g \in S} \int_{(t, \infty)} p_{jg}(t, s) \left(\pi g_t(s) + \sum_{\substack{h \in S \\ h \neq g}} \mu_{gh}(s) \pi g_t^h(s) \right) ds \quad (11.2)$$

where $\pi g_t(s) := \frac{v(s)}{v(t)} \dot{a}_g(s)$ and $\pi g_t^h(s) := \frac{v(s)}{v(t)} a_{gh}(s)$

are the present value at time t of the pension payments $\dot{a}_g(s)$ and benefit payments $a_{gh}(s)$ ($s > t$) with respect to a deterministic technical int. rate $r_t = \delta$ (or $\delta(t)$).

However, what is the prospective reserve $V_{\overline{T}}^+(t, A)$ (with $\overline{T}$ as in Assumption 10.2), if r_t is stochastic?

Using the same reasoning as in Section 10 we find that $V_{\overline{T}}^+(t, A)$ is given by (11.1) and (11.2), where

$\pi g_t(s)$ and $\pi g_t^h(s)$ are the present values at time t of $\dot{a}_g(s)$ and $a_{gh}(s)$ with respect to stochastic interest rates, respectively.

Problem: What is the present value at time t of payments b at maturity s based on stoch. int. rates r_t ?

possible solution: dynamic hedging principle, let i_t ,

we define just as in Section 10 for S_t the present value of a bond claim b by means of replicating portfolios $V_t(\phi)$ composed of a "riskless" asset and a zero-coupon bond.

For this purpose consider:

Bond market model

1. "riskless" asset with price dynamics

$$R(t) = \exp\left(\int_0^t r_s ds\right) \quad (11.3)$$

where r_t is the overnight int.-rate based on market information up to time t , modeled by a $\mathcal{G}_t$ -adapted process.

2. zero-coupon bond with prices

$$P(t, T) > 0, \quad 0 \leq t \leq T \quad (11.4)$$

modeled by a $\mathcal{G}_t$ -adapted proc. $P(t, T), 0 \leq t \leq T$ such that $P(T, T) = 1$ at maturity T .

As a special case of this model we now assume that

$$P(t, T) = P_+(t, r_t) \quad (11.5)$$

for a "nice" function P_+ (i.e., sufficiently regular function)

Remark: or $P(t, T) = P_+(t, r_t, \gamma_t)$ for a proc. $\gamma_t \in \mathbb{R}^{d-1}$.

Further: suppose that r_t satisfies the SDE

$$r_t = r + \int_0^t \mu(u, r_u) du + \int_0^t \sigma(u, r_u) dB_u \quad (11.6)$$

for "nice" functions μ, σ .

Applying Itô's formula to $X_t = r_t$ and $f = P_t$ gives

$$P(t, T) = P(0, T) + \int_0^t \tilde{\mu}(s, r_s) P(s, T) ds + \int_0^t \tilde{\sigma}(s, r_s) P(s, T) dB_s \quad (11.7)$$

where

$$\tilde{\mu}(s, x) := \frac{1}{P_T(s, x)} \left\{ \frac{\partial}{\partial s} P_T(s, x) + \frac{\partial}{\partial x} P_T(s, x) \cdot \mu(s, x) + \frac{1}{2} \frac{\partial^2}{\partial x^2} P_T(s, x) \cdot \sigma^2(s, x) \right\}$$

and

$$\tilde{\sigma}(s, x) := \frac{1}{P_T(s, x)} \left\{ \frac{\partial}{\partial x} P_T(s, x) \cdot \sigma(s, x) \right\} \quad (11.8)$$

Now assume in Girsanov's theorem (Theor. 10.7) a process X_t such that

$$X_t \tilde{\sigma}(s, r_s) = r_s - \tilde{\mu}(s, r_s), \quad s \geq 0 \quad (11.9)$$

then, $\tilde{B}_t := B_t - \int_0^t X_s ds$ is a B.m. under $\tilde{P}$ with

$$\tilde{P}(A) := E[1_A Z_T], \quad Z_t := \exp\left(\int_0^t X_s dB_s - \frac{1}{2} \int_0^t (X_s)^2 ds\right) \quad (11.10)$$

Moreover, see Lih 11, the **discounted bond price**

$$\hat{P}(t, T) := \exp\left(-\int_0^t r_u du\right) P(t, T) \text{ is a } \text{martingale} \text{ under } \tilde{P}.$$

In the same way as in Section 10 (just replace there S_t by $P(t, T)$) the notion of a portfolio

value $V_t(\phi) = \phi_t^0 R(t) + \phi_t^1 P(t, T)$ or of a

replicating strategy $\phi_t = (\phi_t^0, \phi_t^1)$ can be introduced and similarly to Theor. 10.12 one obtains the following result:

Theorem 11.1 (Hedging of bond claims)

Assume that $\tilde{\sigma}(s, r_s) \neq 0$. Further let h be a bond claim with maturity $s < T$ such that

$$E_{\tilde{P}}[(\exp(-\int_0^T r_u du) \cdot h)^2] < +\infty$$

11.4

Then h is replicable and the portfolio value at time t of any replicating strategy ϕ is given by

$$V_t(\phi) = E_{\tilde{P}}[\exp(-\int_t^T r_u du) h | \mathcal{G}_t] \quad (11.12)$$

11.10 $\Rightarrow$ Just as in Section 10, it is reasonable to define the value (or present value) of the bond claim h at time $t \leq T$ as

$$\text{Claim Value}_t = E_{\tilde{P}}[\exp(-\int_t^T r_u du) h | \mathcal{G}_t] \quad (11.12)$$

$$\xrightarrow{t=0} \text{Claim Value}_0 = E_{\tilde{P}}[\exp(-\int_0^T r_u du) h] \quad (11.13)$$

Example 11.2

Choose $h = \dot{a}_g(s)$ or $a_{g,h}(s)$ w.r.t. $V_{\tilde{F}}^+(t, A)$

$$\xrightarrow{11.12} \pi_{\tilde{F}}^g(t, s) = E_{\tilde{P}}[\exp(-\int_t^s r_u du) | \mathcal{G}_t] \dot{a}_g(s) \quad (11.14)$$

$$\pi_{\tilde{F}}^{g,h}(t, s) = \underbrace{E_{\tilde{P}}[\exp(-\int_t^s r_u du) | \mathcal{G}_t]}_{\stackrel{\text{Lit 11}}{=} p(t, s)} \underbrace{a_{g,h}(s)}_{\stackrel{\text{Lit 5}}{=} p_s(t, s+t)} \quad (11.15)$$

$$\text{Then, } V_{\tilde{F}}^+(t, A) = V_j^+(t) |_{j=X_t} = V_j(t, x) |_{j=X_t} \quad (11.16)$$

$$\text{with } V_j(t, x) := \int_t^T C_t^j(u, x) du \quad (11.17)$$

$$\text{where } C_t^j(u, x) := p_u(t, x) \cdot G_u^j(x)$$

$$\text{for } G_u^j(x) := \sum_{i \in S} p_{ji}(t, u) \left\{ \dot{a}_i(u) + \sum_{k \neq i} \mu_{ik}(u) a_{ik}(u) \right\}$$

chain rule
 $\Rightarrow$

$$\frac{\partial}{\partial t} V_j(t, x) = \int_t^T \frac{\partial}{\partial t} C_t^j(u, x) du - \dot{a}_j(t) - \sum_{k \neq j} \mu_{jk}(t) a_{jk}(t) \quad (11.18)$$

Now write

(11.5)

$$d\beta_s = d(\hat{\beta}_s + \int_0^s X_u du) = d\hat{\beta}_s + X_s ds \quad \text{in (11.7) to}$$

get that

$$P(t, T) = P(0, T) + \int_0^t (\hat{\mu}(s, \tau_s) + X_s \hat{\sigma}(s, \tau_s)) P(s, T) ds + \int_0^t \hat{\sigma}(s, \tau_s) P(s, T) d\hat{B}_s \quad (11.12)$$

On the other hand, since

$$P_n(t, x) := C_t^j(u, x) / G_t^j(u) \quad (11.20)$$

and

$$\frac{\partial}{\partial t} p_{ji}(t, u) \stackrel{\text{backward Kolmog.}}{=} \sum_{k \neq j} \mu_{jk}(t) \{ p_{ji}(t, u) - p_{ki}(t, u) \}$$

we get

$$\frac{\partial}{\partial t} P_n(t, x) = \frac{\partial}{\partial t} \left(C_t^j(u, x) / G_t^j(u) \right) \quad (11.21)$$

$$\stackrel{\text{Product rule}}{=} \frac{1}{G_t^j(u)} \left(\sum_{k \neq j} \mu_{jk}(t) \{ C_t^k(u, x) - C_t^j(u, x) \} + \frac{\partial}{\partial t} C_t^j(u, x) \right)$$

(lengthy computations)

Then using (11.8) for $T=u$, the integrand in the drift term of (11.12) equals

$$\begin{aligned} & \frac{1}{G_t^j(u)} \left(\frac{\partial}{\partial t} C_t^j(u, \tau_t) + \sum_{k \neq j} \mu_{jk}(t) \{ C_t^k(u, \tau_t) - C_t^j(u, \tau_t) \} \right. \\ & \left. + \frac{\partial}{\partial x} C_t^j(u, \tau_t) (\mu(t, \tau_t) - X_t \sigma(t, \tau_t)) + \frac{1}{2} \frac{\partial^2}{\partial x^2} C_t^j(u, \tau_t) \sigma(t, \tau_t)^2 \right) \end{aligned} \quad (11.22)$$

But we also know that

$$(\hat{\mu}(t, \tau_t) + X_t \hat{\sigma}(t, \tau_t)) P(t, u) \stackrel{(11.9)}{=} r_t P(t, u) \stackrel{(11.20)}{=} r_t \frac{C_t^j(u, \tau_t)}{G_t^j(u)}$$

$$\text{which implies } r_t \frac{C_t^j(u, \tau_t)}{G_t^j(u)} = (11.22)$$

$$\Rightarrow \lambda_t C_t^j(u, r_t) = (11.22) G_t^j(u) \quad (11.23) \quad \text{11.6}$$

Integrating $\int_t^T (\cdot) du$ on both sides and using (11.17), (11.18) we get

$$\frac{\partial V_j}{\partial t}(t, r_t) = \underbrace{\lambda_t V_j(t, r_t) - \dot{a}_j(t) - \sum_{k \neq j} \mu_{jk}(t) (a_{jk}(t) + V_k^+(t) - V_j^+(t))}_{\text{classical part of Thiele's diff. equation}} \quad (11.29)$$

$$- \left(\frac{\partial}{\partial x} V_j(t, r_t) (\mu(t, r_t) - X_t \sigma(t, r_t)) + \frac{1}{2} \frac{\partial^2}{\partial x^2} V_j(t, r_t) \sigma(t, r_t)^2 \right)$$

→ Thiele's partial diff. equation for stock int. rates

Ex. 11.3 (Vasicek model)

$$\mu(t, x) = a(b - x), \quad \sigma(t, x) = \sigma \quad \text{in (11.6)}$$

Assumption in (11.9): $X_t = -\lambda \in \mathbb{R}$ constant

($\Rightarrow$) λ_t under $\tilde{P}$ in (11.10) is mean-reverting again)

Thiele's part. diff. eqn. for the term insurance:

$$\begin{aligned} \frac{\partial}{\partial t} V_{\#}(t, r_t) &= (\lambda_t + \mu_{\#t}(t)) V_{\#}(t, r_t) - K - \mu_{\#t} a_{\#t}(t) \\ &- \left(\frac{\partial}{\partial x} V_{\#}(t, r_t) (a(b - r_t) - \lambda \sigma) + \frac{1}{2} \frac{\partial^2}{\partial x^2} V_{\#}(t, r_t) \sigma^2 \right) \end{aligned} \quad (11.25)$$