Oblig STK4520
Selection and Dynamics in Mortality Modelling

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May 12, 2014
Outline

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Introduction

When we are dealing with life insurance and pension schemes, which can last over several decades, we must integrate risk of different origins in our calculations. These sources of risk can be:

- Life table risk
- Liability risk due to inflation and discounting
- Financial risk

In this presentation, I will discuss the life table risk, which is connected to the change in life expectancies.
Mortalities since the second world war

![Graph showing life expectancy at birth by region over time.](image-url)

- More developed regions, excluding Eastern Europe
- Latin America and the Caribbean
- Eastern Europe
- Asia
- World
- Developing Oceania
- Africa

**Time period**
- 1950-1955
- 1955-1960
- 1960-1965
- 1965-1970
- 1970-1975
- 1975-1980
- 1980-1985
- 1985-1990
- 1990-1995
- 1995-2000
- 2000-2005
- 2005-2010

**Life expectancy at birth (years)**
- 35
- 40
- 45
- 50
- 55
- 60
- 65
- 70
- 75
- 80
- 85
Consequences of longer lives

- Uncertainty
- Insurance schemes cost more
- Larger reserves required
- New financial laws must be passed
- Must take this into account in actuarial calculations
Notation in life insurance

- $kpl$ is the probability of living to age $l + k$ at age $l$.
- $kql = 1 - kpl$ are the mortality rates.
- $l_0$, $l_r$ and $l_e$ are, in the same order, the age when you enter the scheme, retirement age and realistic maximum age.
- The equivalence premium $\pi$ is the solution of:

$$
\pi \sum_{k=0}^{l_r-l_0-1} \frac{d^k}{d^k} kpl_0 = s \sum_{k=l_r-l_0}^{l_e-l_0} \frac{d^k}{d^k} kpl_0
$$

where $s$ is the pension, and $d = \frac{1}{1+r}$ the discount.
Numerical example of uncertainty in life tables

Both are 100000 simulations
The Lee-Carter model for time-varying mortalities

- The Lee-Carter model (1992) describes dynamic changes in how long people live. The main goal of the model is to predict future mortality for a given population.
- An important prerequisite for the model is that the life expectancy is growing.

**Theorem**

A simplified version of the model takes the mortality of an individual in age \( l \), in year \( k \) as:

\[
q_{lk} = \omega_l^k q_{l0}
\]

where

\[
\log(\omega_l) = -\alpha \frac{e^{gl}}{1 + e^{gl}}
\]

and

\[
g_l = \beta_0 + \beta_1 l + \beta_2 l^2
\]
The reduction factor $\omega^k_l$ when $k = 5, 10$ and $50$

Blue line: $k = 50$
Green line: $k = 10$
Red line: $k = 5$
$\omega^{50}_{20} =$ reduction factor for individual aged 20 in 50 years
Mortalities varying with time in ordinary life-insurance calculations

- First let’s see how mortalities are entered in actuarial calculations

**Operations**

\[
p_l \rightarrow Z \rightarrow \hat{p}_l \rightarrow k\hat{p}_l \rightarrow \hat{\pi}, \hat{X}_k, \hat{PV}_0
\]

- What about mortalities varying over time? We now expect people to live longer:

**Simple model**

- \( q_l(i) = q_{l0}e^{-\gamma(i)} \), the cohort version.
- \( q_{lk} = q_{l0}e^{-\gamma_k} \), the time dynamic version.
- \( \gamma(i) \) and \( \gamma_k \) are parameters that makes the expected mortalities deviate from the default sequence \( q_{l0} \)
k-step survival probabilities

- \( k+1p_l = (1 - q_{l+k}(l))kp_l \), cohort version.
- \( k+1p_l = (1 - q_{l+k,k})kp_l \), time dynamic version.

Now, inserting the simple models in the k-step recursion gives us:

Life tables

- \( k+1p_l = (1 - q_0e^{-\gamma l})kp_l \), cohort version.
- \( k+1p_l = (1 - q_0e^{-\gamma k})kp_l \), time dynamic version.

We can now compute the entire life table \( kp_l \) with \( k = 1, \ldots \), and all the operations that were explained above can be done.
Present value of pension portfolios

After finding an appropriate life table model, we can find the present value of the liabilities, reserve and premiums.

Some notation

\[
\hat{PV}_0 = \sum_{k=0}^{l_e-l_0} d^k \hat{X}_k
\]

where

\[
\hat{X}_k = -\hat{\pi} \sum_{l=l_0}^{l_{l_r-k}} J_{lk} \hat{p}_l + s \sum_{l=l_{l_r-k}}^{l_{e-k}} J_{lk} \hat{p}_l
\]
Results

<table>
<thead>
<tr>
<th>Premium</th>
<th>Present Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\pi} = 9.162$ (unit: thousand)</td>
<td>$PV_0 = 30.961$ (unit: billion)</td>
</tr>
</tbody>
</table>

**Table:** Premium and present value for the non parametric model

<table>
<thead>
<tr>
<th>Added years</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mortality parameter</td>
<td>$\gamma = 0$</td>
<td>$\gamma = 0.077$</td>
<td>$\gamma = 0.155$</td>
<td>$\gamma = 0.233$</td>
<td>$\gamma = 0.312$</td>
<td>$\gamma = 0.391$</td>
</tr>
<tr>
<td>Net reserve (billion)</td>
<td>30.7</td>
<td>32.1</td>
<td>33.5</td>
<td>34.9</td>
<td>36.3</td>
<td>37.8</td>
</tr>
</tbody>
</table>

**Table:** Net reserve when people live longer than expected

<table>
<thead>
<tr>
<th>Added longevity (years)</th>
<th>0</th>
<th>2.5</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Net reserve (billion)</td>
<td>30.7</td>
<td>32.0</td>
<td>33.4</td>
</tr>
</tbody>
</table>

**Table:** Net reserve under the time dynamic model
One time premia

- First let’s see the one time premia for the non-parametric and the parametric mortality models.
Now let’s see the Flat Cut and the dynamic models.

For the Flat Cut we introduce \( \zeta_l \), such that \( q'_l = \zeta_l q_l \), a 15 percent cut gives \( \zeta_l = 0.85 \).

The dynamic model is the Lee-Carter model as explained earlier.
Flat Cut versus Dynamic model

![Graph comparing Flat Cut and Dynamic models](image)

**Age**

- Flat Cut
- Dynamic
A selection model

- Mortalities of people who are insured or members of pension schemes are often much lower than for the average populations.
- In a cross-country study, Mitchell and McCarthy (2002) found mortalities to be as much as 25 percent lower for life-insurance clients.

Proposition

Let

\[ q'_l = \zeta_l q_l \]

through

\[ \zeta_l = (1 + e^{-g_l})^{-1} \]

where

\[ g_l = \beta_0 + \beta_1 (l - 20) + \beta_2 (l - 20)^2 \]
Let’s compare this model to the other models we saw earlier, the estimates we will use (which are for a European company) are: $\hat{\beta}_0 = 1.340$, $\hat{\beta}_1 = -0.099$ and $\hat{\beta}_2 = 0.00235$. 

![One-time premia](image1.png)

![Mortality selection factor](image2.png)
The End

Thank you for your attention.