UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: STK4530 — SOLUTIONS

Day of examination: December 5, 2018

Examination hours: 09.00 – 14.00

This problem set consists of 5 pages.

Appendices: None

Permitted aids: Godkjent kalkulator

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

1a

By definition of the dynamics,

$$r(T) = r(t) + \int_t^T \mu(u)du + \int_t^T \sigma(u)d\widetilde{W}(u)\,, 0 \leq t \leq T$$

Using $T = s \geq t$ in the above, we find

$$\begin{aligned}
\int_t^T r(s)ds &= \int_t^T r(t) + \int_t^s \mu(u)du + \int_t^s \sigma(u)d\widetilde{W}(u)\,ds \\
&= (T-t)r(t) + \int_t^T \int_u^T ds\mu(u)du + \int_t^T \int_u^T du\sigma(u)d\widetilde{W}(u) \\
&= (T-t)r(t) + \int_t^T (T-u)\mu(u)du + \int_t^T (T-u)\sigma(u)d\widetilde{W}(u)
\end{aligned}$$

In the second equality we used Fubini-Tonelli and the stochastic Fubini theorem.

1b

$\int_t^T (T-u)\sigma(u)d\widetilde{W}(u)$ is a Gaussian random variable with mean zero and variance $\int_t^T (T-u)^2\sigma^2(u)du$, since σ is deterministic. Thus, $\int_t^T r(s)ds$ is Gaussian, and it is known that a Gaussian random variable has finite exponential moments, i.e., $\exp(-\int_t^T r(s)ds)$ has finite expectation.

We have that $r(t)$ is $\mathcal{F}_t$-measurable, $\int_t^T (T-u)\mu(u)du$ is deterministic,

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and thus

$$\begin{aligned}
P(t,T) &= \mathbb{E}_Q\left[\exp(-\int_t^T r(s)ds \,|\, \mathcal{F}_t\right] \\
&= \exp\left(-(T-t)r(t) - \int_t^T (T-u)\mu(u)du\right)\mathbb{E}_Q\left[\exp(-\int_t^T (T-u)\sigma(u)d\widetilde{W}(u)) \,|\, \mathcal{F}_t\right] \\
&= \exp\left(-(T-t)r(t) - \int_t^T (T-u)\mu(u)du\right)\mathbb{E}_Q\left[\exp(-\int_t^T (T-u)\sigma(u)d\widetilde{W}(u))\right] \\
&= \exp\left(-(T-t)r(t) - \int_t^T (T-u)\mu(u)du + \frac{1}{2}\int_t^T (T-u)^2\sigma^2(u)du\right)
\end{aligned}$$

In the third equality we used the independent increment property of the stochastic integral when the integrand is deterministic, and in the fourth equality we applied the general rule for the expectated value of an exponential of a Gaussian variable with zero mean.

1c

Money market account is $B(t) = \exp(\int_0^t r(s)ds)$. Thus, since $B(t)$ is $\mathcal{F}_t$-measurable, we find

$$P(t,T)B^{-1}(t) = \mathbb{E}_Q\left[\exp(-\int_0^T r(s)ds) \,|\, \mathcal{F}_t\right]$$

From b) above we know that $\exp(-\int_0^T r(s)ds)$ has finite expectation, and therefore also $P(t,T)B^{-1}(t)$ has finite expectation (even positive). By double conditioning, we find for $t \geq u$

$$\mathbb{E}_Q\left[P(t,T)B^{-1}(t) \,|\, \mathcal{F}_u\right] = \mathbb{E}_Q\left[\exp(-\int_0^T r(s)ds) \,|\, \mathcal{F}_u\right] = P(u,T)B^{-1}(u)$$

Hence, Q-martingale.

We notice that $P(t,T)B^{-1}(t) = \exp(-(T-t)r(t) + v(t,T) - \int_0^t r(s)ds)$ for some explicity given deterministic function v. Thus, to fnd the dynamics of $P(t,T)B^{-1}(t)$, we can appeal to Ito's Formula on the function $g(t,r) = \exp(-(T-t)r + v(t,T) + \int_0^t r(s)ds)$. In Ito's formula, we obtain "dt-terms" from $\partial_t g$, $\partial_{rr} g$. We obtain a "dr-term" from $\partial_r g$. Indeed, this term is

$$\partial_r g(t,r(t))dr(t) = -(T-t)g(t,r(t))(\mu(t)dt + \sigma(t)d\widetilde{W}(t))$$

As we know that $P(t,T)B^{-1}(t)$ is a Q-martingale, all the "dt-terms" must add up to zero, and we are thus left with

$$d(P(t,T)/B(t)) = -(T-t)\sigma(t)(P(t,T)/B(t))d\widetilde{W}(t)$$

Problem 2

2a

We find from the Radon Nikodym derivative (on $\mathcal{F}_T$),

$$\mathbb{E}_{Q^T}[X] = \mathbb{E}_Q\left[X\frac{dQ^T}{dQ}\right] = \mathbb{E}_Q\left[X\frac{P(T,T)}{P(0,T)B(T)}\right] = P(0,T)^{-1}\mathbb{E}_Q\left[\exp(-\int_0^T r(s)ds)X\right]$$

since $P(T,T) = 1$.

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2b

The no-arbitrage drift condition of f tells us that

$$df(t,T)=\alpha(t,T)dt+\sigma d\widetilde{W}(u)$$

where $\alpha(t,T)=\sigma\int_t^T\sigma du=\sigma^2(T-t)$. Hence,

$$df(t,T)=\sigma^2(T-t)dt+\sigma d\widetilde{W}(u)$$

or,

$$f(t,T)=f(0,T)+\sigma^2(tT-\frac{1}{2}t^2)+\sigma\widetilde{W}(t)$$

Since $r(t)=f(t,t,)$, we find immediately that

$$r(t)=f(0,t)+\frac{1}{2}\sigma^2t^2+\sigma\widetilde{W}(t)$$

From the definition of Q^T and $P(t,T)=\exp(-\int_t^T f(t,u)du)$, we find

$$\begin{aligned}\frac{dQ^T}{dQ}\Big|_{\mathcal{F}_t}&=\frac{P(t,T)}{P(0,T)B(t)}\\&=\frac{\exp(-\int_t^T f(t,u)ds)}{\exp(-\int_0^T f(0,u)du)\exp(\int_0^t r(u)du)}\\&=\exp\left(\int_0^T f(0,u)-\int_t^T f(t,u)du-\int_0^t r(u)du\right)\end{aligned}$$

By using the dynamics of $f(t,u)$ found above, we calculate

$$\begin{aligned}\int_t^T f(t,u)du&=\int_t^T f(0,u)du+\sigma^2\int_t^T(tu-\frac{1}{2}t^2)du+\sigma\int_t^T\widetilde{W}(t)du\\&=\int_t^T f(0,u)du+\frac{1}{2}\sigma^2tT(T-t)+\sigma(T-t)\widetilde{W}(t)\end{aligned}$$

Moreover,

$$\begin{aligned}\int_0^t r(u)du&=\int_0^t f(0,u)du+\int_0^t\frac{1}{2}\sigma^2u^2du+\int_0^t\sigma\widetilde{W}(u)du\\&=\int_0^t f(0,u)du+\frac{1}{6}\sigma^2t^3+\sigma\int_0^t\int_0^u d\widetilde{W}(s)du\\&=\int_0^t f(0,u)du+\frac{1}{6}\sigma^2t^3+\sigma\int_0^t\int_s^t dud\widetilde{W}(s)\\&=\int_0^t f(0,u)du+\frac{1}{6}\sigma^2t^3+\sigma\int_0^t(t-s)d\widetilde{W}(s)\end{aligned}$$

In the third equality we used the stochastic Fubini theorem. We now collect these terms together, noticing that $(T-t)\widetilde{W}(t)+\int_0^t(t-s)d\widetilde{W}(s)=\int_0^t(T-s)d\widetilde{W}(s)$ and $\int_0^t(T-s)^2ds=tT^2-Tt^2+t^3/3$. This shows the desired result.

(Continued on page 4.)

2c

Girsanov tells us that dQ^T/dQ is a measure change yielding a probability Q^T for which

$$dW^T(t) = d\widetilde{W}(t) + \sigma(T-t)dt$$

is a Q^T-Brownian motion for $t \leq T$. Inserting this into the dynamics of f yields,

$$df(t,T) = \sigma^2(T-t)dt + \sigma(dW^T(t) - \sigma(T-t)dt) = \sigma dW^T(t)$$

Thus,

$$f(t,T) = f(0,T) + \sigma W^T(t)$$

From a) and general theory of pricing, we find that the zero-price is

$$\begin{aligned}\pi(0) &= P(0,T)\mathbb{E}_{Q^T}[\mathbf{1}(r(T) \leq K)] = P(0,T)\mathbb{E}_{Q^T}[\mathbf{1}(f(T,T) \leq K)] \\ &= P(0,T)Q^T(f(0,T) + \sigma W^T(T) \leq K)\end{aligned}$$

But $W^T(T)$ is a normally distributed random variable with mean zero and variance T, so

$$\pi(0) = P(0,T)\Phi\left(\frac{K - f(0,T)}{\sigma\sqrt{T}}\right)$$

with Φ being the cumulative standard normal distribution.

Problem 3

3a

From Ito's Formula we find

$$L(t,T_2) = L(0,T_2)\exp\left(\lambda_2 W^{T_3}(t) - \frac{1}{2}\lambda_2^2 t\right)$$

This is $\mathcal{F}_t$-adapted stochastic process. We know that W^{T_3} is Gaussian, and a Gaussian variable has finite exponential moments. Thus, $L(t,T_2)$ has finite expectation for all $t \leq T_2$. By the independent increment property of the Brownian motion W^{T_3} and the $\mathcal{F}_s$-measurability of $W^{T_3}(s)$, we find for $s \leq t \leq T_2$

$$\begin{aligned}\mathbb{E}_{Q^{T_3}}[L(t,T_2)|\mathcal{F}_s] &= L(0,T_2)e^{-\frac{1}{2}\lambda_2^2 t}\mathbb{E}_{Q^{T_3}}[e^{\lambda_2 W^{T_3}(t)}|\mathcal{F}_s] \\ &= L(0,T_2)e^{-\frac{1}{2}\lambda_2^2 t}\mathbb{E}_{Q^{T_3}}[e^{\lambda_2(W^{T_3}(t)-W^{T_3}(s))}]e^{\lambda_2 W^{T_3}(s)} \\ &= L(s,T_2)\end{aligned}$$

In the last equality, we used the rule for exponential moments of mean-zero Gaussian variables. This shows the martingale property of $t \mapsto L(t,T_2)$.

As already indicated, since $W^{T_3}(t)$ is mean-zero normally distributed with variance t, we have that $\ln L(t,T_2) = \ln L(0,T_2) + \lambda W^{T_3}(t) - \frac{1}{2}\lambda_2^2 t$ is a normal random variable with variance $\lambda_2^2 t$ and mean $\ln L(0,T_2) - \lambda_2^2 t/2$. Hence, $L(t,T_2)$ is lognormal.

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3b

By definition, $t \mapsto \mathcal{E}_t(\sigma_{T_2,T_3} \bullet W^{T_3})$ is a local Q^{T_3}-martingale whenever σ_{T_2,T_3} is an Ito integrable process. But $L(t, T_2)$ is positive adapted stochastic process, and λ_2 a constant, so by definition σ_{T_2,T_3} is Ito integrable as it becomes bounded and positive and adapted. Moreover, by the boundedness, Novikov's condition is satisfied:

$$\sigma_{T_2,T_3}(t) \leq \lambda_2$$

thus,

$$\mathbb{E}_{Q^{T_3}}[e^{\frac{1}{2}\int_0^{T_2} \sigma^2_{T_2,T_3}(s)ds}] \leq e^{\frac{1}{2}\lambda_2^2 T_2} < \infty.$$

Therefore we can conclude that $t \mapsto \mathcal{E}_t(\sigma_{T_2,T_3} \bullet W^{T_3})$ is a true Q^{T_3}-martingale. As the mean of this martingale is 1, it follows from Girsanov's Theorem that $Q^{T_2} \sim Q^{T_3}$ and W^{T_2} is a Q^{T_2}-Brownian motion.

The Q^{T_2}-dynamics of $L(t, T_2)$ is

$$\begin{aligned} dL(t, T_2) &= \lambda_2 L(t, T_2)(dW^{T_2}(t) - \sigma_{T_2,T_3}(t)dt) \\ &= \frac{\delta L^2(t, T_2)}{\delta L(t, T_2) + 1}\lambda_2^2 dt + \lambda_2 L(t, T_2)dW^{T_2}(t) \end{aligned}$$

We see that the drift is stochastic, and therefore the lognormal property is lost under Q^{T_2}.

3c

From above, we define the Q^{T_2}-dynamics of $L(t, T_1)$, $t \leq T_1$ as

$$dL(t, T_1) = \lambda_1 L(t, T_1)dW^{T_2}(t)$$

The price of the T_2-claim X is

$$\pi(0) = P(0, T_2)\mathbb{E}_{Q^{T_2}}[1(L(T_1, T_1) \leq \kappa)]$$

We have that

$$L(T_1, T_1) = L(0, T_1) \exp\left(\lambda_1 W^{T_2}(T_1) - \frac{1}{2}\lambda_1^2 T_1^2\right)$$

Hence,

$$\pi(0) = P(0, T_2)Q^{T_2}\left(\ln L(0, T_1) - \frac{1}{2}\lambda_1^2 T_1^2 + \lambda_1 W^{T_2}(T_1) \leq \ln \kappa\right)$$

or,

$$\pi(0) = P(0, T_2)\Phi(d)$$

where Φ is the cumulative standard normal distribution and

$$d = \frac{\ln(\kappa/L(0, T_2)) + \frac{1}{2}\lambda_1^2 T_1^2}{\lambda_1\sqrt{T_1}}$$

END