Non-life insurance mathematics (STK4540)

Solutions to the problems of the 1. mandatory assignment

Problem 1 (i) By assumption the claim sizes $X_i, i \geq 1$ are *i.i.d.* with common distribution $X_1 \sim Exp(\lambda)$. So by Exercises 1, Problem 4 we know that the MLE $\widehat{\lambda}$ is given by

$$\widehat{\lambda} = \frac{n}{\sum_{i=1}^{n} X_i}.$$

The observed fire losses are $X_1 = 1.877063, X_2 = 1.268152, ..., X_{12} = 1.990099 + 1.987624$, $X_{13} = 1.155941$, $X_{14} = 3.378713$. So $n = 14$ (sample size) and we get that

$$\widehat{\lambda} = 0.426182.$$

(ii) If the data set was large, the graph of the QQ-plot would indicate that the real claim size distribution is heavier tailed than the assumed one, since it curves down to the right.

(iii) The theoretical mean excess function is given by $e_F(u) = \frac{1}{\widehat{\lambda}} \approx 2.346$ for all u (see Exercises 2, Problem 1). If the data set was large, the graph which is curving up towards the end wouldn't indicate convergence against that level and rather suggest that the real claim size distribution is more heavy-tailed than the assumed one.

(iv) We know from Prob. 1, Exerc. 1 that $S := \sum_{i=1}^{n} X_i \stackrel{d}{=} \Gamma(n, \lambda)$ for $X_i, i \geq 1$ i.i.d with $X_i \stackrel{d}{=} Exp(\lambda)$ and that

$$P(S > x) = \exp(-\lambda x) \sum_{j=0}^{n-1} \frac{(\lambda x)^j}{j!} \text{ (Erlang distribution).}$$

So for $n = 5$, $\lambda = \widehat{\lambda} = 0.426182$ and $x = 10$ we get

$$P(S > 10) \approx 0.578.$$

Problem 2 (i)

$$\overline{F}(x) = x^{-\alpha} L(x),$$

where

$$L(x) := \left(\frac{x}{x+1}\right)^{\alpha} (1 + \alpha \log(x+1)), x > 0.$$

Since

$$\begin{aligned} & \frac{\left(\frac{\lambda x}{\lambda x+1}\right)^{\alpha}}{\left(\frac{x}{x+1}\right)^{\alpha}} \frac{(1 + \alpha \log(\lambda(x+1)))}{(1 + \alpha \log(x+1))} \\ = & \frac{\left(\frac{\lambda}{\lambda+\frac{1}{x}}\right)^{\alpha}}{\left(\frac{1}{1+\frac{1}{x}}\right)^{\alpha}} \frac{(1 + \alpha \log(\lambda) + \alpha \log(x+1))}{(1 + \alpha \log(x))} \underset{x \longrightarrow \infty}{\longrightarrow} 1 \end{aligned}$$

for all $\lambda > 0$, it follows that L is slowly varying. Therefore X_1 is regularly varying with index $\alpha > 0$ (see Def. 3.4.4 in the lecture notes).

Hence, by Prop. 3.4.6 and Remark 3.4.8 X_1 is subexponential, too.

(ii) By differentiating the distribution function F we can compute its probability density and obtain the following log-likelihood function with respect to the claim sizes

$$l(\alpha) = 2n\log(\alpha) + \sum_{i=1}^{n}(-(\alpha+1)\log(x_i) + \log\log(x_i)).$$

By solving the equation

$$\frac{d}{d\alpha}l(\alpha) = \frac{2n}{\alpha} - \sum_{i=1}^{n}\log(x_i) = 0$$

we see that the maximum of the likelihood function is attained at

$$\widehat{\alpha} = \frac{2n}{\sum_{i=1}^{n}\log(x_i)} \overset{n=14}{\approx} 3.01.$$

Problem 3 We know that

$$\Psi(u) \sim \rho^{-1}\overline{F}^*_{X_1,I}(u)$$

for $u \longrightarrow \infty$ (Th. 3.4.13).

We want to approximate

$$F^*_{X_1,I}(u) = \frac{1}{E^*[X_1]}\int_0^u P^*(X_1 > y)dy$$

by means of the empirical distribution function F_n.

So we see that

$$E^*[X_1] \approx \frac{1}{n}\sum_{i=1}^{n} X_i \overset{n=14}{=} 2.34642.$$

On the other hand,

$$P^*(X_1 > y) \approx 1 - F_n(y).$$

Hence,

$$\begin{aligned}\int_0^u P^*(X_1 &> y)dy \approx \int_0^u (1 - \frac{1}{n}\sum_{i=1}^{n} 1_{(-\infty,y]}(X_i))dy \\ &= u - \frac{1}{n}\sum_{i=1}^{n}(\max(u, X_i) - X_i) \\ &= u - \frac{1}{n}\sum_{i=1}^{n}(u - \min(u, X_i))\end{aligned}$$

Using the latter, we find that $\Psi(4) = 0.521$ and $\Psi(7) = 0.065$.

Problem 4 See the self-explaining hint.

Problem 5 Our assumptions imply that

$$E[\overline{S}(t)] = E[\overline{N}(t)]E[X_1].$$

In order to compute $E[\overline{N}(t)]$ we want to employ "dobbelforventning" and the order statistics property of the Poisson process (see e.g. the book of T. Mikosch). More precisely, let us denote by E_1 the expectation w.r.t. the mixing variable $\theta \sim Exp(1)$ and E_2 the expectation w.r.t. a standard homogeneous Poisson process $\widetilde{N}(t)$. So we find that

$$E[\overline{N}(t)] = E_1[E_2[\sum_{j=1}^{\widetilde{N}(\theta t)} \mathbf{1}_{[0,t]}(T_j + V_j)]].$$

From non-life insurance or more precisely from e.g. Prop. 2.1.16 in the book of T. Mikosch (2nd edition, Springer) we know that

$$E_2[\sum_{j=1}^{\widetilde{N}(\theta t)} \mathbf{1}_{[0,t]}(T_j + V_j)] = E_2[\sum_{j=1}^{\widetilde{N}(\theta t)} \mathbf{1}_{[0,t]}(tU_j + V_j)],$$

where the $i.i.d$ uniformly distributed random variables U_j, $j \geq 1$ are independent of V_j, $\widetilde{N}(t)$ and X_j, $j \geq 1$. Further we observe that

$$E_2[\sum_{j=1}^{\widetilde{N}(\theta t)} \mathbf{1}_{[0,t]}(tU_j + V_j)] = E_2[\widetilde{N}(\theta t)]E_2[\mathbf{1}_{[0,t]}(tU_1 + V_1)].$$

So it follows that

$$\begin{aligned} E[\overline{N}(t)] &= E_1[E_2[\widetilde{N}(\theta t)]]E_2[\mathbf{1}_{[0,t]}(tU_1 + V_1)] \\ &= E_1[\theta t]E_2[\mathbf{1}_{[0,t]}(tU_1 + V_1)] \\ &= tE_2[\mathbf{1}_{[0,t]}(tU_1 + V_1)]. \end{aligned}$$

On the other hand we obtain

$$\begin{aligned} & E_2[\mathbf{1}_{[0,t]}(tU_1 + V_1)] \\ = & \int_0^1 \int_0^1 \mathbf{1}_{[0,t]}(tx_1 + x_2)dx_1dx_2 \\ = & \int_{1\wedge t}^1 \int_0^1 \mathbf{1}_{[0,t]}(tx_1 + x_2)dx_1dx_2 + \int_0^{1\wedge t} \int_0^1 \mathbf{1}_{[0,t]}(tx_1 + x_2)dx_1dx_2 \\ & 0 + \int_0^{1\wedge t} \int_0^1 \mathbf{1}_{[0,t]}(tx_1 + x_2)dx_1dx_2 \\ & \overset{x_1=\frac{1}{t}(u-x_2)}{=} \int_0^{1\wedge t} \frac{1}{t} \int_{x_2}^{t+x_2} \mathbf{1}_{[0,t]}(u)dudx_2 \\ = & \int_0^{1\wedge t} \frac{1}{t} \int_{x_2}^{t} 1dudx_2 = \int_0^{1\wedge t} \frac{1}{t}(t - x_2)dx_2 \\ = & 1 \wedge t - \frac{1}{t}(\frac{x_2^2}{2}\Big|_{x_2=0}^{1\wedge t}) = 1 \wedge t - \frac{1}{t}\frac{(1\wedge t)^2}{2}. \end{aligned}$$

Hence

$$E[\overline{S}(t)] = t(1 \wedge t - \frac{1}{t}\frac{(1 \wedge t)^2}{2})E[X_1].$$

It follows that

$$E[S(t)] = tE[X_1] > 2E[\overline{S}(t)] \text{ for } 0 \leq t < 1$$

and

$$E[S(t)] \approx E[\overline{S}(t)]$$

in the long run.

Problem 6 It can be shown as in Problem 7 of the second mandatory assignment that $S(t), t \geq 0$ has independent and stationary increments. Using the latter property, we find for $0 < t_1 < t_2 < ... < t_n$ that

$$\begin{aligned}
& E\left[\exp(-\lambda_1 S(t_1) - ... - \lambda_n S(t_n))\right] \\
= \; & E\left[\exp(-\lambda_1 S(t_1) - \lambda_2(S(t_1) + (S(t_2) - S(t_1))) - ... - \lambda_n(S(t_1) + (S(t_2) - S(t_1)) + ... + (S(t_n) - S(t_{n-1})\right. \\
= \; & E\left[\exp(-(\lambda_1 + ... + \lambda_n)S(t_1) - (\lambda_2 + ... + \lambda_n)(S(t_2) - S(t_1)) - ... - \lambda_n(S(t_n) - S(t_{n-1})))\right] \\
= \; & E\left[\exp(-(\lambda_1 + ... + \lambda_n)S(t_1))\right] \cdot E\left[\exp(-(\lambda_2 + ... + \lambda_n)(S(t_2) - S(t_1)))\right] \cdot ... \cdot E\left[\exp(-\lambda_n(S(t_n) - S(t_{n-1}\right. \\
= \; & E\left[\exp(-(\lambda_1 + ... + \lambda_n)S(t_1))\right] \cdot E\left[\exp(-(\lambda_2 + ... + \lambda_n)(S(t_2 - t_1)))\right] \cdot ... \cdot E\left[\exp(-\lambda_n(S(t_n - t_{n-1})))\right].
\end{aligned}$$

On the other hand, using monotone convergence, we get that

$$\begin{aligned}
& E\left[\exp(-\lambda_1 S(t_1))\right] \\
= \; & E\left[\exp(-\lambda_1 \sum_{i=1}^{N(t_1)} (X_i - x)_+)\right] \\
= \; & E\left[\exp(-\lambda_1 \sum_{i=1}^{N(t_1)} (X_i - x)_+)(\sum_{k \geq 0} 1_{\{N(t_1)=k\}})\right] \\
= \; & \sum_{k \geq 0} E\left[\exp(-\lambda_1 \sum_{i=1}^{k} (X_i - x)_+) 1_{\{N(t_1)=k\}}\right] \\
= \; & \sum_{k \geq 0} E\left[\exp(-\lambda_1 \sum_{i=1}^{k} (X_i - x)_+)\right] P(N(t_1) = k) \\
= \; & \sum_{k \geq 0} (E\left[\exp(-\lambda_1 (X_1 - x)_+)\right])^k \frac{(\lambda t_1)^k}{k!} e^{-\lambda t_1}.
\end{aligned}$$

By applying the probability density of the exponential distribution, we also find that

$$\begin{aligned}
& E\left[\exp(-\lambda_1(X_1-x)_+)\right] \\
= \;& E\left[1_{\{X_1\le x\}}\exp(-\lambda_1\cdot 0)\right] + E\left[1_{\{X_1>x\}}\exp(-\lambda_1(X_1-x))\right] \\
= \;& P(X_1\le x) + e^{\lambda_1 x}E\left[1_{\{X_1>x\}}\exp(-\lambda_1 X_1)\right] \\
= \;& 1-e^{-\gamma x} + e^{\lambda_1 x}\int_x^\infty \exp(-\lambda_1 y)\gamma\exp(-\gamma y)dy \\
= \;& 1-e^{-\gamma x} + \gamma e^{\lambda_1 x}\int_x^\infty \exp(-(\lambda_1+\gamma)y)dy \\
= \;& 1-e^{-\gamma x} + \gamma e^{\lambda_1 x}\left(-\frac{1}{(\lambda_1+\gamma)}\exp(-(\lambda_1+\gamma)y)\Big|_{y=x}^\infty\right) \\
= \;& 1-e^{-\gamma x} + \gamma e^{\lambda_1 x}\frac{1}{(\lambda_1+\gamma)}\exp(-(\lambda_1+\gamma)x) \\
= \;& 1-e^{-\gamma x} + \frac{\gamma}{(\lambda_1+\gamma)}e^{-\gamma x} \\
= \;& 1+(\frac{\gamma}{(\lambda_1+\gamma)}-1)e^{-\gamma x}.
\end{aligned}$$

So

$$\begin{aligned}
& E\left[\exp(-\lambda_1 S(t_1))\right] \\
= \;& \sum_{k\ge 0}(1+(\frac{\gamma}{(\lambda_1+\gamma)}-1)e^{-\gamma x})^k\frac{(\lambda t_1)^k}{k!}e^{-\lambda t_1} \\
= \;& \exp\left((1+(\frac{\gamma}{(\lambda_1+\gamma)}-1)e^{-\gamma x})\lambda t_1-\lambda t_1\right) \\
= \;& \exp\left((\frac{\gamma}{(\lambda_1+\gamma)}-1)e^{-\gamma x}\lambda t_1\right).
\end{aligned}$$

On the other hand, consider the total claim amount process

$$\widetilde{S}(t)=\sum_{i=1}^{N^*(t)}X_i,$$

where $N^*(t), t\ge 0$ is a homogeneous Poisson process with intensity $\lambda^*>0$, which is independent of the claim sizes $X_i, i\ge 1$. Then we can carry out the same calculations as above for $x=0$ and obtain that

$$\begin{aligned}
& E\left[\exp(-\lambda_1\widetilde{S}(t_1))\right] \\
= \;& \exp\left((\frac{\gamma}{(\lambda_1+\gamma)}-1)\lambda^* t_1\right).
\end{aligned}$$

Choose now $\lambda^*=\lambda e^{-\gamma x}$. Then we see that

$$E\left[\exp(-\lambda_1\widetilde{S}(t_1))\right]=E\left[\exp(-\lambda_1 S(t_1))\right]$$

for all $\lambda_1, t_1 > 0$. Since also $\widetilde{S}(t), t \geq 0$ is a process with independent and stationary increments, it follows from (1) and the hint that

$$(\widetilde{S}(t_1), ..., \widetilde{S}(t_n)) \stackrel{\text{law}}{=} (S(t_1), ..., S(t_n))$$

for all $0 < t_1 < t_2 < ... < t_n$. The latter (in connection with the Laplace-Stieltjes transform) also implies that

$$(\widetilde{U}(t_1), ..., \widetilde{U}(t_n)) \stackrel{\text{law}}{=} (U(t_1), ..., U(t_n))$$

for all $0 < t_1 < t_2 < ... < t_n$, where $\widetilde{U}(t) := u + ct - \widetilde{S}(t)$ is another risk process. Hence,

$$\Psi(u) = \widetilde{\Psi}(u),$$

where $\widetilde{\Psi}(u)$ is the ruin probability with respect to $\widetilde{U}(t), t \geq 0$ (which also satisfies the Net Profit condition for $\gamma \geq 1$).

We also know from Problem 3, Exercises 2 that

$$\widetilde{\Psi}(u) = \frac{1}{1+\rho} \exp(-\gamma \frac{\rho}{1+\rho} u).$$

So for $u = 20$, $\gamma = \lambda = 1$, $x = 15$ and $\rho = 0.2$ we get that $\Psi(20) = 0.0297$.

Problem 7 We want to show that the tail distribution $\overline{F} = 1 - F$ of the Zeta distribution is regularly varying with index $\alpha - 1 > 0$, that is

$$\lim_{x \longrightarrow \infty} \frac{1 - F(\lambda x)}{1 - F(x)} = \lambda^{-(\alpha-1)}$$

for all $\lambda > 0$.

By definition we have that

$$F(x) = P(X \leq x) = \sum_{\substack{k \leq x, \\ k \in \mathbb{N}}} \frac{1}{\zeta(\alpha) k^\alpha} = \sum_{k \leq [x]} \frac{1}{\zeta(\alpha) k^\alpha},$$

where $[x]$ denotes the Gauss bracket of x, that is $[x] := \max\{n \in \mathbb{Z} : n \leq x\}$.

So for $x \geq 1$ we get that

$$\begin{aligned} F(x) &= \frac{1}{\zeta(\alpha)} \int_1^{[x]+1} [y]^{-\alpha} dy = \frac{1}{\zeta(\alpha)} \sum_{k=1}^{[x]} \int_k^{k+1} [y]^{-\alpha} dy \\ &= \frac{1}{\zeta(\alpha)} \sum_{k=1}^{[x]} \int_k^{k+1} k^{-\alpha} dy = \frac{1}{\zeta(\alpha)} \sum_{k=1}^{[x]} k^{-\alpha}. \end{aligned}$$

On the other hand, we see for $\lambda y > 1$ that

$$(\frac{y-1}{\lambda y})^\alpha \leq \frac{[\lambda y]^{-\alpha}}{[y]^{-\alpha}} = (\frac{[y]}{[\lambda y]})^\alpha \leq (\frac{y}{\lambda y - 1})^\alpha,$$

since $x-1 \leq [x] \leq x$ for all x.

We also observe that

$$\lim_{y \longrightarrow \infty} (\frac{y-1}{\lambda y})^{\alpha} = \lim_{y \longrightarrow \infty} (\frac{y}{\lambda y - 1})^{\alpha} = \lambda^{-\alpha}.$$

So

$$\lim_{y \longrightarrow \infty} \frac{[\lambda y]^{-\alpha}}{[y]^{-\alpha}} = \lambda^{-\alpha},$$

that is the function f given by $f(x) = [x]^{-\alpha}, x \geq 1$ is regularly varying with index $-\alpha$. Then it follows from the theorem of Karamata (Exercises 3, Problem 4) that

$$\frac{\int_x^{\infty} [y]^{-\alpha} dy}{x[x]^{-\alpha}} \underset{x \longrightarrow \infty}{\longrightarrow} -(1-\alpha)^{-1}. \tag{2}$$

Therefore, we obtain that

$$\begin{aligned} \frac{1-F(\lambda x)}{1-F(x)} &= \frac{\int_{[\lambda x]+1}^{\infty} f(y)dy}{\int_{[x]+1}^{\infty} f(y)dy} \\ &= \frac{\frac{\int_{[\lambda x]+1}^{\infty} f(y)dy}{([\lambda x]+1)f([\lambda x]+1)}}{\frac{\int_{[x]+1}^{\infty} f(y)dy}{\frac{([\lambda x]+1)}{([x]+1)} \frac{f([\lambda x]+1)}{f([x]+1)} ([x]+1)f([x]+1)}} \underset{x \longrightarrow \infty}{\longrightarrow} \lambda^{-(\alpha-1)} \end{aligned}$$

because of (2) and

$$\frac{([\lambda x]+1)}{([x]+1)} \underset{x \longrightarrow \infty}{\longrightarrow} \lambda$$

as well as

$$\frac{f([\lambda x]+1)}{f([x]+1)} = (\frac{[x]+1}{[\lambda x]+1})^{\alpha} \underset{x \longrightarrow \infty}{\longrightarrow} \lambda^{-\alpha}.$$

Problem 8 (i) Define the process

$$Y_t = 1 + \sum_{i=-\infty}^{t-1} \prod_{j=i+1}^{t} A_j, t \in \mathbb{Z}.$$

We want to show that $Y_t, t \in \mathbb{Z}$ solves the recurrence equation

$$Y_t = A_t Y_{t-1} + 1, t \in \mathbb{Z}, \tag{3}$$

and is strictly stationary.

We see that

$$\begin{aligned} A_t Y_{t-1} + 1 &= A_t(1 + \sum_{i=-\infty}^{t-2} \prod_{j=i+1}^{t-1} A_j) + 1 \\ &= A_t + \sum_{i=-\infty}^{t-2} \prod_{j=i+1}^{t} A_j + 1 \\ &= \sum_{i=t-1}^{t-1} \prod_{j=i+1}^{t} A_j + \sum_{i=-\infty}^{t-2} \prod_{j=i+1}^{t} A_j + 1 \\ &= \sum_{i=-\infty}^{t-1} \prod_{j=i+1}^{t} A_j + 1 = Y_t \end{aligned}$$

for all t.

Next, we have to check that the sum in the definition of Y_t converges with probability 1. For this purpose, let us choose a $c \in (0,1)$ such that $E[\log A_1] < \log c < 0$ (since by our assumption $E[\log A_1] < 0$).

On the other hand, the *i.i.d.*-property of A_j and the SLLN imply that

$$|t-i|^{-1} \sum_{j=i+1}^{t} \log A_j \underset{i \longrightarrow -\infty}{\longrightarrow} E[\log A_1]$$

with probability 1, that is for a sure event Ω^*. Hence, for all $\omega \in \Omega^*$ there exists a $i_0(\omega)$ such that for all $i \leq i_0(\omega)$

$$|t-i|^{-1} \sum_{j=i+1}^{t} \log A_j < \log c.$$

So

$$\prod_{j=i+1}^{t} A_j = \exp(|t-i|\,(|t-i|^{-1} \sum_{j=i+1}^{t} \log A_j)) \leq \exp(|t-i| \log c) = c^{|t-i|}$$

for all $i \leq i_0(\omega)$. Therefore it follows for all $-m \leq i_0(\omega)$ $(\leq t-1)$ that

$$\begin{aligned} \sum_{i=-m}^{t-1} \prod_{j=i+1}^{t} A_j &= \sum_{i=i_0(\omega)-1}^{t-1} \prod_{j=i+1}^{t} A_j + \sum_{i=-m}^{i_0(\omega)} \prod_{j=i+1}^{t} A_j \leq \sum_{i=i_0(\omega)-1}^{t-1} \prod_{j=i+1}^{t} A_j + \sum_{i=-m}^{i_0(\omega)} c^{|t-i|} \\ &\overset{\text{geometric series}}{\leq} \sum_{i=i_0(\omega)-1}^{t-1} \prod_{j=i+1}^{t} A_j + \sum_{r \geq 0} c^r \\ &= \sum_{i=i_0(\omega)-1}^{t-1} \prod_{j=i+1}^{t} A_j + \frac{1}{1-c} < \infty. \end{aligned}$$

So $\sum_{i=-\infty}^{t-1} \prod_{j=i+1}^{t} A_j$ converges with probability 1.

Next, we aim at verifying strict stationarity.

Claim: For all $n \geq 1$ and $m \leq t$ and $h \in \mathbb{Z}$ we have that

$$((A_s)_{m\leq s\leq t}, ..., (A_s)_{m\leq s\leq t+n-1}) \stackrel{d}{=} ((A_s)_{m+h\leq s\leq t+h}, ..., (A_s)_{m+h\leq s\leq t+h+n-1}). \tag{4}$$

Let us w.l.o.g. show this for $n = 2$ and $h > 0$: In this case the characteristic function of the left hand side of (4) is given by

$$\begin{aligned}
& E[\exp(i(\lambda_1 A_m + ... + \lambda_{t-m+1} A_t + \mu_1 A_1 + ... + \mu_{t-m+2} A_{t+1})] \\
= \; & E[\exp(i(\lambda_1 + \mu_1) A_m)]...E[\exp(i(\lambda_{t-m+1} + \mu_{t-m+1} A_t)]E[\exp(i\mu_{t-m+2} A_{t+1})] \\
= \; & E[\exp(i(\lambda_1 + \mu_1) A_{m+h})]...E[\exp(i(\lambda_{t-m+1} + \mu_{t-m+1}) A_{t+h})]E[\exp(i\mu_{t-m+2} A_{t+h+1})] \\
= \; & E[\exp(i(\lambda_1 A_{m+h} + ... + \lambda_{t-m+1} A_{t+h} + \mu_1 A_1 + ... + \mu_{t-m+2} A_{t+h+1})]
\end{aligned}$$

for all λ_i, μ_j because of the *i.i.d.*- property of $A_j, j \in \mathbb{Z}$. So (4) holds.

It follows from (4) that

$$\begin{aligned}
(Y_t^m, Y_{t+1}^m) \quad : \quad & = (\sum_{i=-m}^{t-1} \prod_{j=i+1}^{t} A_j, \sum_{i=-m}^{t} \prod_{j=i+1}^{t+1} A_j) \\
& \stackrel{d}{=} (\sum_{i=-m}^{t-1} \prod_{j=i+1}^{t} A_{j+h}, \sum_{i=-m}^{t} \prod_{j=i+1}^{t+1} A_{j+h}) \\
& = (\sum_{i=-m+h}^{t+h-1} \prod_{j=i+1}^{t+h} A_j, \sum_{i=-m+h}^{t+h} \prod_{j=i+1}^{t+h+1} A_j) \\
& \underset{m\longrightarrow\infty}{\longrightarrow} (Y_{t+h}, Y_{t+h+1})
\end{aligned}$$

with probability 1. On the other hand, we know that

$$(Y_t^m, Y_{t+1}^m) \underset{m\longrightarrow\infty}{\longrightarrow} (Y_t, Y_{t+1})$$

with probability 1. Hence,

$$(Y_t, Y_{t+1}) \stackrel{d}{=} (Y_{t+h}, Y_{t+h+1})$$

or (w.l.o.g.) more generally

$$(Y_t, ..., Y_{t+n-1}) \stackrel{d}{=} (Y_{t+h}, ..., Y_{t+h+n-1}), \tag{5}$$

which gives strict stationarity.

In the next step we want to show uniqueness (in the class of strictly stationary solutions): Assume another strictly stationary solution $Y_t^{'}, t \in \mathbb{Z}$. Then $Y_t - Y_t^{'}$ solves the equation

$$Z_t = A_t Z_{t-1}.$$

Iteration of this equation entails that

$$Y_t - Y_t^{'} = A_t ... A_{t-i+1}(Y_{t-i} - Y_{t-i}^{'})$$

for all $i \geq 1$. So

$$\left|Y_t - Y_t^{'}\right| = A_t...A_{t-i+1}\left|Y_{t-i} - Y_{t-i}^{'}\right|.$$

Using the SLLN, we can show as before that

$$A_t...A_{t-i+1} \underset{i\longrightarrow\infty}{\longrightarrow} 0$$

with probability 1. On the other hand by using (4) and a similar argument as in the case of the proof of strict stationarity we obtain that

$$A_t...A_{t-i+1}Y_{t-i} \overset{d}{=} A_{t+i}...A_{t+1}Y_t.$$

Here again

$$A_{t+i}...A_{t+1} \underset{i\longrightarrow\infty}{\longrightarrow} 0$$

with probability 1. So for all $\varepsilon > 0$ we have that

$$P(A_t...A_{t-i+1}Y_{t-i} > \varepsilon) = P(A_{t+i}...A_{t+1}Y_t > \varepsilon) \underset{i\longrightarrow\infty}{\longrightarrow} 0.$$

So

$$A_t...A_{t-i+1}Y_{t-i} \underset{i\longrightarrow\infty}{\longrightarrow} 0$$

in probability. By assumption we know that $Y_{t-i}^{'} \overset{d}{=} Y_t^{'}$ for all i. So

$$Y_{t-i}^{'} \underset{i\longrightarrow\infty}{\longrightarrow} Y_t^{'}$$

in distribution. Since $A_{t+i}...A_{t+1} \underset{i\longrightarrow\infty}{\longrightarrow} 0$ with probability 1 it follows that

$$A_t...A_{t-i+1}Y_{t-i}^{'} \underset{i\longrightarrow\infty}{\longrightarrow} 0$$

in distribution. Using the latter argument once more, we obtain that

$$A_t...A_{t-i+1}(Y_{t-i} - Y_{t-i}^{'}) \underset{i\longrightarrow\infty}{\longrightarrow} 0$$

in distribution. Since $|\cdot|$ is continuous, we see that

$$\left|Y_t - Y_t^{'}\right| = A_t...A_{t-i+1}\left|Y_{t-i} - Y_{t-i}^{'}\right| \underset{i\longrightarrow\infty}{\longrightarrow} 0$$

in distribution. Thus for all $\varepsilon > 0$

$$P(\left|Y_t - Y_t^{'}\right| > \varepsilon) = 1 - P(\left|Y_t - Y_t^{'}\right| \leq \varepsilon) = 0.$$

So

$$Y_t = Y_t^{'}$$

with probability 1, which yields uniqueness.

(ii) Because of strict stationarity we know that

$$Y_t \stackrel{d}{=} Y_0 = 1 + \sum_{i=-\infty}^{-1} \prod_{j=i+1}^{0} A_j.$$

On the other hand, since

$$(A_{-m+1}, ..., A_0) \stackrel{d}{=} (A_1, ..., A_m)$$

we see that

$$\begin{aligned} \sum_{i=-m}^{-1} \prod_{j=i+1}^{0} A_j &\stackrel{d}{=} \sum_{i=-m}^{-1} \prod_{j=i+1}^{0} A_{-j+1} \\ = \sum_{i=1}^{m} \prod_{j=1}^{i} A_j. \end{aligned}$$

So letting $m \longrightarrow \infty$, we get that

$$Y_t \stackrel{d}{=} Y_0 \stackrel{d}{=} \sum_{i \geq 1} \prod_{j=1}^{i} A_j.$$

Therefore we have for $x > 0$ that

$$\begin{aligned} P(Y_t > x) = P(Y_0 > x) &\geq P(\sup_{n \geq 1} \prod_{j=1}^{n} A_j > x) \\ &= P(\sup_{n \geq 1} \sum_{j=1}^{n} \log A_j > \log x) \\ &= P(\sup_{n \geq 1} S_n > \log x), \end{aligned}$$

where

$$S_n := \sum_{j=1}^{n} Z_j$$

for the *i.i.d.* random variables $Z_j := \log A_j$ such that $E[Z_1] < 0$, which corresponds to a NPC condition in ruin theory.

Then we can employ the proof of Theorem 3.2.3 (Lundberg's inequality) to the partial sum S_n and we get for $x \geq 1$ that

$$m(x) := P(\sup_{n \geq 1} S_n > \log x) \leq \exp(-r \log x) = x^{-r}.$$