Non-Life Insurance Mathematics (STK4540)
Solutions to the 2.mandatory assignment

Problem 1 We know from Theorem 6.3.1 in Mikosch that

$$\widehat{\mu}_{LB} = (1-w)\mu + w\overline{X},$$

where

$$w = \frac{n\lambda}{\varphi + n\lambda} \text{ and } \overline{Y} = \frac{1}{n}\sum_{j=1}^{n} Y_j.$$

By Exercises 5, Prob. 2 we have that

1.

$$\begin{aligned} \mu &= E[p(\theta)] = E[P(X_1 > K\,|\theta)] = E[(\frac{\lambda'}{K})^{\theta}] \\ &= \left(\frac{\beta}{\beta - \log(\lambda'/K)}\right)^{\gamma} \approx 0.18806. \end{aligned}$$

2.

$$\begin{aligned} \lambda &= Var[p(\theta)] = Var[(\frac{\lambda'}{K})^{\theta}] \\ &= \left(\frac{\beta}{\beta - 2\log(\lambda'/K)}\right)^{\gamma} - \left(\frac{\beta}{\beta - \log(\lambda'/K)}\right)^{2\gamma} \approx 0.06842. \end{aligned}$$

3.

$$\begin{aligned} \varphi &= E[Var[Y_1\,|\theta]] = E[p(\theta)] - E[(p(\theta))^2] \\ &= \left(\frac{\beta}{\beta - \log(\lambda'/K)}\right)^{\gamma} - \left(\frac{\beta}{\beta - 2\log(\lambda'/K)}\right)^{\gamma} \approx 0.08427. \end{aligned}$$

So

$$w \overset{n=10}{=} \frac{10 \cdot 0.06842}{0.08427 + 10 \cdot 0.06842} \approx 0.89034 \text{ and } \overline{Y} = 0.1.$$

Therefore, we get that

$$\widehat{\mu}_{LB} = (1-w)\mu + w\overline{Y} \approx 0.109$$

estimated probability for $X_1 > 75000$ NOK given θ.

The corresponding risk is given by

$$\rho(\widehat{\mu}_{LB}) = (1-w)\lambda \approx 0.0075.$$

Problem 2 (i) As in Problem 1 we want to use Theorem 6.3.1 in Mikosch and obtain that

$$\widehat{\mu}_{LB} = (1-w)\mu + w\overline{X},$$

where

$$w = \frac{n\lambda}{\varphi + n\lambda} \text{ and } \overline{X} = \frac{1}{n}\sum_{j=1}^{n} X_j.$$

Here:

1.
$$\mu = E[\mu(\theta)] = E[\theta] \overset{\text{hint}}{=} \frac{2.3}{1.2} = 1.91667.$$

2.
$$\lambda = Var[\mu(\theta)] = Var[\theta] \overset{\text{hint}}{=} \frac{2.3}{(1.2)^2} = 1.59722.$$

3.
$$\varphi = E[Var[X_1\,|\theta]] = E[\theta] \overset{\text{hint}}{=} \frac{2.3}{1.2} = 1.91667.$$

Thus

$$w \overset{n=10}{=} \frac{10 \cdot 1.59722}{1.91667 + 10 \cdot 1.59722} = 0.892857 \text{ and } \overline{X} = 0.9.$$

Therefore, we get that

$$\begin{aligned} \widehat{\mu}_{LB} &= (1-w)\mu + w\overline{X} \\ &\approx 1.00893 \end{aligned}$$

expected claim number (per year) given the observations.

The corresponding risk is given by

$$\rho(\widehat{\mu}_{LB}) = (1-w)\lambda \approx 0.171131.$$

(ii) We now assume in (i) that $\theta = \exp(Z)$ with $Z \sim \mathcal{N}(0,1)$. Then

1.
$$\mu = E[\mu(\theta)] = E[\exp(Z)] = e^{\frac{1}{2}} \approx 1.649.$$

2.
$$\begin{aligned} \lambda &= Var[\mu(\theta)] = E[(\mu(\theta))^2] - \mu^2 \\ &= E[e^{2Z}] - \mu^2 = e^{\frac{1}{2}\cdot 4} - \mu^2 \\ &\approx 4.670. \end{aligned}$$

3.
$$\varphi = E[Var[X_1\,|\theta]] = E[\theta] \approx 1.649.$$

So

$$w \overset{n=10}{=} \frac{10 \cdot 4.67}{1.649 + 10 \cdot 4.67} \approx 0.966 \text{ and } \overline{X} = 0.9.$$

Hence

$$\widehat{\mu}_{LB} = (1-w)\mu + w\overline{X} \approx 0.925.$$

Problem 3 Once more we see from Theorem 6.3.1 in Mikosch that

$$\widehat{\mu}_{LB} = (1-w)\mu + w\overline{X},$$

where

$$w = \frac{n\lambda}{\varphi + n\lambda} \text{ and } \overline{X} = \frac{1}{n}\sum_{j=1}^{n} X_j.$$

Here:

1.

$$\begin{aligned}
\mu &= E[\mu(\theta)] = E[E[\exp(\theta + \tau Z_1)\,|\theta]] \\
&= E[\exp(\theta)E[\exp(\tau Z_1)\,|\theta]] \\
&\overset{\theta, Z_1 \text{ independent}}{=} E[\exp(\theta)E[\exp(\tau Z_1)]] \\
&\overset{\text{hint in Ex4, Prob.4}}{=} E[\exp(\theta)\exp(\frac{1}{2}\tau^2)] \\
&\overset{\text{hint in Ex4, Prob.4}}{=} \exp(\mu' + \frac{1}{2}(\tau^2 + \sigma^2)) \\
&= e.
\end{aligned}$$

2.

$$\begin{aligned}
\lambda &= Var[\mu(\theta)] = E[(\mu(\theta))^2] - \mu^2 \\
&= E[\exp(2\theta)(E[\exp(\tau Z_1)])^2] - \mu^2 \\
&= \exp(2\mu' + \tau^2 + 2\sigma^2) - \mu^2 \\
&= e^3 - e^2.
\end{aligned}$$

3.

$$\begin{aligned}
\varphi &= E[Var[X_1\,|\theta]] = E[E[X_1^2\,|\theta]] - E[(E[X_1\,|\theta])^2] \\
&= E[E[\exp(2\theta + 2\tau Z_1)\,|\theta]] - E[\exp(2\theta)\exp(\tau^2)] \\
&= E[\exp(2\theta)E[\exp(2\tau Z_1)]] - E[\exp(2\theta)\exp(\tau^2)] \\
&= E[\exp(2\theta)\exp(2\tau^2)] - E[\exp(2\theta)\exp(\tau^2)] \\
&= \exp(2\mu' + 2\sigma^2 + 2\tau^2) - \exp(2\mu' + 2\sigma^2 + \tau^2) \\
&= e^4 - e^3
\end{aligned}$$

So

$$\begin{aligned}
w &= \frac{e^2 8(e-1)}{e^2(e^2-e) + e^2 8(e-1)} \\
&= \frac{8(e-1)}{(e^2-e) + 8(e-1)} \approx 0.746
\end{aligned}$$

and

$$\begin{aligned}
\widehat{\mu}_{LB} &= (1-w)\mu + w\overline{X} \\
&= 0.254 \cdot e + 0.746 \cdot 4.5 \approx 4.05 \text{ (in 1000 NOK)}.
\end{aligned}$$

Problem 4 Suppose that $E[X_1] < \infty$. Then for $X = X_1$ in Problem 1, we obtain that

$$\begin{aligned} E[X_1] &= \int_0^\infty P(X_1 \geq z)dz = \int_0^\infty E[P(X_1 \geq z\,|\theta)]dz \\ &= \int_0^\lambda E[P(X_1 \geq z\,|\theta)]dz + \int_\lambda^\infty E[P(X_1 \geq z\,|\theta)]dz \\ &= \int_0^\lambda 1dz + + \int_\lambda^\infty E[(\frac{\lambda}{z})^\theta]dz \\ &\overset{n\geq\lambda}{\geq} \int_\lambda^n E[(\frac{\lambda}{z})^\theta]dz \end{aligned} \tag{1}$$

On the other hand,

$$E[(\frac{\lambda}{z})^\theta] = \int_0^\infty \lambda^y z^{-y} f_{\Gamma(\gamma,\beta)}(y)dy.$$

So it follows from (1) by inter-changing integrals (Tonelli's theorem) that

$$\begin{aligned} E[X_1] &\geq \int_0^{1/2} \lambda^y f_{\Gamma(\gamma,\beta)}(y) \int_\lambda^n z^{-y} dz dy \\ &= \int_0^{1/2} \lambda^y f_{\Gamma(\gamma,\beta)}(y) \frac{1}{1-y}(n^{1-y} - \lambda^{1-y})dy \\ &\geq -2\int_0^{1/2} \lambda^y f_{\Gamma(\gamma,\beta)}(y)\lambda^{1-y}dy \\ &\quad + n^{1-\frac{1}{2}} \int_0^{1/2} \lambda^y f_{\Gamma(\gamma,\beta)}(y)\frac{1}{1-y}dy \\ &\quad \underset{n\longrightarrow\infty}{\longrightarrow} \infty, \end{aligned}$$

which leads to a contradiction. So $E[X] = \infty$.

Problem 5 (i) It follows from Lemma 5.2.3 (1) in Mikosch that

$$f_\theta(y\,|X = k) = \frac{f_\theta(y)P(X = k\,|\theta = y)}{P(X = k)},$$

where

$$P(X = k) = E[P(X = k\,|\theta)] = E[e^{-\theta}\frac{\theta^k}{k!}] = \int_0^\infty e^{-y}\frac{y^k}{k!}f_\theta(y)dy$$

and

$$P(X = k\,|\theta = y) = e^{-y}\frac{y^k}{k!}.$$

(ii) So by (i) we get that

$$\begin{aligned} m_k &= E[\theta \,|X=k] = \int_0^\infty y \frac{f_\theta(y) e^{-y} \frac{y^k}{k!}}{P(X=k)} dy \\ &= \frac{(k+1)!}{k! P(X=k)} \int_0^\infty e^{-y} \frac{y^{k+1}}{(k+1)!} f_\theta(y) dy \\ &= \frac{(k+1)}{P(X=k)} E[e^{-\theta} \frac{\theta^{k+1}}{(k+1)!}] \\ &= \frac{(k+1)}{P(X=k)} E[P(X=k+1\,|\theta)] \\ &= \frac{(k+1)}{P(X=k)} P(X=k+1) \end{aligned}$$

(iii) Consider w.l.o.g. the case $l=2$:

$$\begin{aligned} & E[\theta^2 \,|X=k] \overset{(i)}{=} \int_0^\infty y^2 \frac{f_\theta(y) e^{-y} \frac{y^k}{k!}}{P(X=k)} dy \\ = & \frac{(k+2)!}{k! P(X=k)} \int_0^\infty e^{-y} \frac{y^{k+2}}{(k+2)!} f_\theta(y) dy \\ = & \frac{(k+1)(k+2)}{P(X=k)} E[P(X=k+2\,|\theta)] \\ = & (k+1) \frac{P(X=k+1)}{P(X=k)} (k+2) \frac{P(X=k+2)}{P(X=k+1)} \\ = & m_k m_{k+1}. \end{aligned}$$

Problem 6 (i) We know from Lemma 5.2.3 that

$$f_\theta(y\,|X=k) = \frac{f_\theta(y) P(X_1=x_1\,|\theta=y) ... P(X_1=x_n\,|\theta=y)}{P(X=(x_1,...,x_n)')},$$

where

$$f_\theta(y) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} y^{a-1}(1-y)^{b-1}, 0<y<1$$

and

$$P(X_1=x_j\,|\theta=y) = \left(\begin{array}{c} k \\ x_j \end{array} \right) y^{x_j}(1-y)^{k-x_j} \text{ (cond. binomial distr.)},$$

$j=1,...,n, x_j=0,...,k.$

We find that

$$\begin{aligned}
P(X &= (x_1,...,x_n)') = \\
P(X_1 &= x_1,...,X_n = x_n) = \\
E[P(X_1 &= x_1,...,X_n = x_n\,|\theta)] \overset{X_j, j\geq 1 \text{ i.i.d. given } \theta}{=} \\
E[P(X_1 &= x_1\,|\theta)...P(X_1 = x_n\,|\theta)] = \\
&E[\prod_{j=1}^{n}\left(\begin{array}{c} k \\ x_j \end{array}\right)\theta^{x_j}(1-\theta)^{k-x_j}] \\
&= \int_0^1 \prod_{j=1}^{n}\left(\begin{array}{c} k \\ x_j \end{array}\right) y^{x_j}(1-y)^{k-x_j} f_\theta(y)dy \\
&= \int_0^1 \prod_{j=1}^{n}\left(\begin{array}{c} k \\ x_j \end{array}\right) y^{x_j}(1-y)^{k-x_j} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} y^{a-1}(1-y)^{b-1}dy \\
&= \{\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\prod_{j=1}^{n}\left(\begin{array}{c} k \\ x_j \end{array}\right)\} \times \\
&\quad \times \int_0^1 y^{(a+\sum_{j=1}^n x_j)-1}(1-y)^{(b+nk-\sum_{j=1}^n x_j)-1}\frac{\Gamma(a'+b')}{\Gamma(a')\Gamma(b')}dy\frac{\Gamma(a')\Gamma(b')}{\Gamma(a'+b')} \\
&= \{\frac{\Gamma(a')\Gamma(b')}{\Gamma(a'+b')}\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\prod_{j=1}^{n}\left(\begin{array}{c} k \\ x_j \end{array}\right)\}\int_0^1 y^{a'-1}(1-y)^{b'-1}\frac{\Gamma(a'+b')}{\Gamma(a')\Gamma(b')}dy \\
&= \frac{\Gamma(a')\Gamma(b')}{\Gamma(a'+b')}\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\prod_{j=1}^{n}\left(\begin{array}{c} k \\ x_j \end{array}\right),
\end{aligned}$$

where

$$a' := a + \sum_{j=1}^{n} x_j, b' := b + nk - \sum_{j=1}^{n} x_j > 0.$$

Hence,

$$\begin{aligned}
f_\theta(y\,|X = k) &= \frac{f_\theta(y)P(X_1 = x_1\,|\theta = y)...P(X_1 = x_n\,|\theta = y)}{P(X = (x_1,...,x_n)')} \\
&= \frac{\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}y^{a-1}(1-y)^{b-1}\prod_{j=1}^{n}\left(\begin{array}{c} k \\ x_j \end{array}\right)y^{x_j}(1-y)^{k-x_j}}{\frac{\Gamma(a')\Gamma(b')}{\Gamma(a'+b')}\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\prod_{j=1}^{n}\left(\begin{array}{c} k \\ x_j \end{array}\right)} \\
&= \frac{\Gamma(a'+b')}{\Gamma(a')\Gamma(b')}y^{a'-1}(1-y)^{b'-1},
\end{aligned}$$

which is the density of the Beta distribution for the parameters $a', b' > 0$.

(ii)

$$\widehat{\mu}_B = E[\mu(\theta)\,|X_1,...,X_n] = (E[\mu(\theta)\,|X_1 = x_1,...,X_n = x_n])|_{x_1=X_1,...,x_n=X_n}.$$

On the other hand,

$$\begin{aligned}
& E[\mu(\theta)\,|X_1 = x_1, ..., X_n = x_n] \\
= & \int_0^1 E[X_1\,|\theta = y]\, f_\theta(y\,|X_1 = x_1, ..., X_n = x_n)\, dy \\
& \underset{\text{expectation of a binomial r.v.}}{=} \\
& \int_0^1 (ky)\frac{\Gamma(a'+b')}{\Gamma(a')\Gamma(b')} y^{a'-1}(1-y)^{b'-1} dy \\
= & k\frac{\Gamma(a'+b')}{\Gamma(a')\Gamma(b')} \int_0^1 (y^{(a+1)-1}(1-y)^{b'-1}\frac{\Gamma(a''+b')}{\Gamma(a'')\Gamma(b')} dy \frac{\Gamma(a'')\Gamma(b')}{\Gamma(a''+b')} \\
= & k\frac{\Gamma(a'+b')}{\Gamma(a')\Gamma(b')}\frac{\Gamma(a'')\Gamma(b')}{\Gamma(a''+b')} \\
& \underset{\Gamma(z+1)=z\Gamma(z),\Gamma(1)=1}{=} \\
& k\frac{a'}{a'+b'} \\
= & k\frac{a+\sum_{j=1}^n x_j}{a+b+nk}
\end{aligned}$$

where $a'' := a' + 1$.

As for the corresponding risk we know from Theorem 5.2.1 in Mikosch that

$$\rho(\widehat{\mu}_B) = E[Var[\mu(\theta)\,|X_1, ..., X_n]].$$

Since X_1 is binomially distributed given θ, we see that

$$\mu(\theta) = E[X_1\,|\theta] = k\theta.$$

So

$$\rho(\widehat{\mu}_B) = k^2 E[Var[\theta\,|X_1, ..., X_n]].$$

From the hint and the fact that the conditional density of θ is that of a Beta distribution with parameters $a', b' > 0$, we get

$$\begin{aligned}
& Var[\theta\,|X_1 = x_1, ..., X_n = x_n] \\
= & a'b'/[(a'+b'+1)(a'+b')^2] \\
= & (a+\sum_{j=1}^n x_j)(b+nk-\sum_{j=1}^n x_j)/[(a+b+nk+1)(a+b+nk)^2].
\end{aligned}$$

Therefore, the latter implies that

$$\begin{aligned}
\rho(\widehat{\mu}_B) = & \frac{k^2}{[(a+b+nk+1)(a+b+nk)^2]} E[(a+\sum_{j=1}^n X_j)(b+nk-\sum_{j=1}^n X_j)] \\
= & \frac{k^2}{[(a+b+nk+1)(a+b+nk)^2]} \\
& \times (a(b+nk) + (b-a)E[\sum_{j=1}^n X_j] - E[(\sum_{j=1}^n X_j)^2]).
\end{aligned}$$

Further,

$$E[\sum_{j=1}^{n} X_j] = nE[E[X_1\,|\theta]] = nE[k\theta] \stackrel{\text{hint}}{=} nk\frac{a}{a+b}$$

and

$$\begin{aligned}
& E[(\sum_{j=1}^{n} X_j)^2] \\
= & \sum_{i,j=1}^{n} E[E[X_i X_j\,|\theta]] \\
= & \sum_{i,j=1,i\neq j}^{n} E[E[X_i\,|\theta]\,E[X_j\,|\theta]] + \sum_{i=1}^{n} E[E[X_i^2\,|\theta]] \\
= & (n^2-n)E[(E[X_1\,|\theta])^2] + nE[Var[X_1\,|\theta] + (E[X_1\,|\theta])^2] \\
= & n^2E[(E[X_1\,|\theta])^2] + nE[Var[X_1\,|\theta]] \\
= & n^2E[(k\theta)^2] + nE[k\theta(1-\theta)] \\
= & (n^2k^2-nk)E[\theta^2] + nkE[\theta] \\
= & (n^2k^2-nk)(Var[\theta] + (E[\theta])^2) + nkE[\theta] \\
& \stackrel{\text{hint}}{=} (n^2k^2-nk)(\{ab/[(a+b+1)(a+b)^2]\} + (\frac{a}{a+b})^2) \\
& +nk\frac{a}{a+b}.
\end{aligned}$$

So

$$\begin{aligned}
& \rho(\widehat{\mu}_B) \\
= & \frac{k^2}{[(a+b+nk+1)(a+b+nk)^2]} \\
& \times(a(b+nk) + (b-a)nk\frac{a}{a+b} \\
& -(n^2k^2-nk)(\{ab/[(a+b+1)(a+b)^2]\} + (\frac{a}{a+b})^2) \\
& -nk\frac{a}{a+b}).
\end{aligned}$$

Problem 7 (i) We have that

$$S(t) - S(s) = \sum_{i=N(s)+1}^{N(t)} X_i.$$

Then it follows from the independence of $N(t), t \geq 0$ of the *i.i.d.* claims $X_j, j \geq 1$ that

$$\begin{aligned}
P(S(t)-S(s) &\leq x, S(s) \leq y) = E\left[\mathbf{1}_{\{S(t)-S(s)\leq x\}}\mathbf{1}_{\{S(s)\leq y\}}\right] = E\left[\mathbf{1}_{\left\{\sum_{i=N(s)+1}^{N(t)} X_i \leq x\right\}}\mathbf{1}_{\left\{\sum_{i=1}^{N(s)} X_i \leq y\right\}}\right] \\
&= E\left[\sum_{m\geq 0}\sum_{l=0}^{m}\mathbf{1}_{\{N(t)=m\}}\mathbf{1}_{\{N(s)=l\}}\mathbf{1}_{\left\{\sum_{i=l+1}^{N(t)} X_i \leq x\right\}}\mathbf{1}_{\left\{\sum_{i=1}^{N(s)} X_i \leq y\right\}}\right] \\
&= \sum_{m\geq 0}\sum_{l=0}^{m} E\left[\mathbf{1}_{\{N(t)=m\}}\mathbf{1}_{\{N(s)=l\}}\mathbf{1}_{\left\{\sum_{i=l+1}^{m} X_i \leq x\right\}}\mathbf{1}_{\left\{\sum_{i=1}^{l} X_i \leq y\right\}}\right] \\
&= \sum_{m\geq 0}\sum_{l=0}^{m} E\left[\mathbf{1}_{\{N(t)=m\}}\mathbf{1}_{\{N(s)=l\}}\right] E\left[\mathbf{1}_{\left\{\sum_{i=l+1}^{m} X_i \leq x\right\}}\mathbf{1}_{\left\{\sum_{i=1}^{l} X_i \leq y\right\}}\right] \\
&= \sum_{m\geq 0}\sum_{l=0}^{m} E\left[\mathbf{1}_{\{N(t)=m\}}\mathbf{1}_{\{N(s)=l\}}\right] E\left[\mathbf{1}_{\left\{\sum_{i=l+1}^{m} X_i \leq x\right\}}\right] E\left[\mathbf{1}_{\left\{\sum_{i=1}^{l} X_i \leq y\right\}}\right] \\
&= \sum_{m\geq 0}\sum_{l=0}^{m} E\left[\mathbf{1}_{\{N(t)=m\}}\mathbf{1}_{\{N(s)=l\}}\right] E\left[\mathbf{1}_{\left\{\sum_{i=1}^{m-l} X_i \leq x\right\}}\right] E\left[\mathbf{1}_{\left\{\sum_{i=1}^{l} X_i \leq y\right\}}\right] \\
&= E\left[\mathbf{1}_{\left\{\sum_{i=1}^{N(t)-N(s)} X_i \leq x\right\}}\right] E\left[\mathbf{1}_{\left\{\sum_{i=1}^{N(s)} X_i \leq y\right\}}\right].
\end{aligned}$$

However,

$$\begin{aligned}
&E\left[\mathbf{1}_{\left\{\sum_{i=1}^{N(t)-N(s)} X_i \leq x\right\}}\right] \\
&= \sum_{m\geq 0}\sum_{l=0}^{m} E\left[\mathbf{1}_{\{N(t)=m\}}\mathbf{1}_{\{N(s)=l\}}\mathbf{1}_{\left\{\sum_{i=1}^{m-l} X_i \leq x\right\}}\right] \\
&= \sum_{m\geq 0}\sum_{l=0}^{m} E\left[\mathbf{1}_{\{N(t)=m\}}\mathbf{1}_{\{N(s)=l\}}\right] E\left[\mathbf{1}_{\left\{\sum_{i=1}^{m-l} X_i \leq x\right\}}\right] \\
&= \sum_{m\geq 0}\sum_{l=0}^{m} E\left[\mathbf{1}_{\{N(t)=m\}}\mathbf{1}_{\{N(s)=l\}}\right] E\left[\mathbf{1}_{\left\{\sum_{i=l+1}^{m} X_i \leq x\right\}}\right] \\
&= E\left[\mathbf{1}_{\left\{\sum_{i=N(s)+1}^{N(t)} X_i \leq x\right\}}\right] = P(\sum_{i=N(s)+1}^{N(t)} X_i \leq x) \\
&= P(S(t)-S(s) \leq x).
\end{aligned}$$

Hence,

$$P(S(t)-S(s) \leq x, S(s) \leq y) = P(S(t)-S(s) \leq x)P(S(s) \leq y)$$

for all $x, y \in \mathbb{R}$. So $S(t)-S(s)$ is independent of $S(s)$ for $s < t$.

We also see from the first equation by letting $y \longrightarrow \infty$ that

$$P(S(t)-S(s) \leq x) = P(\sum_{i=1}^{N(t)-N(s)} X_i \leq x).$$

But then $N(t)-N(s) \stackrel{d}{=} N(t-s)$ (stationarity of the Poisson process) gives

$$\begin{aligned} P(S(t)-S(s) &\leq x) = P(\sum_{i=1}^{N(t)-N(s)} X_i \leq x) \\ &= \sum_{m\geq 0} E\left[\mathbf{1}_{\{N(t)-N(s)=m\}}\right] E\left[\mathbf{1}_{\left\{\sum_{i=1}^m X_i \leq x\right\}}\right] \\ &= \sum_{m\geq 0} E\left[\mathbf{1}_{\{N(t-s)=m\}}\right] E\left[\mathbf{1}_{\left\{\sum_{i=1}^m X_i \leq x\right\}}\right] \\ &= \sum_{m\geq 0} E\left[\mathbf{1}_{\{N(t-s)=m\}}\mathbf{1}_{\left\{\sum_{i=1}^m X_i \leq x\right\}}\right] \\ &= E\left[\mathbf{1}_{\left\{\sum_{i=1}^{N(t-s)} X_i \leq x\right\}}\right] = P(S(t-s)\leq x). \end{aligned} \tag{2}$$

So $S(t), t \geq 0$ has independent and stationary increments, that is $S(t), t \geq 0$ is a special example of a *Lévy process*.

(ii) It follows from the properties of conditional expectations, the independence of $S(t) - S(s)$ and $S(s)$ and the stationarity of $S(t), t \geq 0$ that

$$\begin{aligned} & E\left[\exp(-hU(t))|\ S(s)\right] \\ = \ & E\left[\exp(-h(u+ct)+hS(t))|\ S(s)\right] \\ = \ & \exp(-h(u+ct))E\left[\exp(h(S(t)-S(s)+S(s)))|\ S(s)\right] \\ = \ & \exp(-h(u+ct))E\left[\exp(hS(s))\exp(h(S(t)-S(s)))|\ S(s)\right] \\ = \ & \exp(-h(u+ct))\exp(hS(s))E\left[\exp(h(S(t)-S(s)))|\ S(s)\right] \\ = \ & \exp(-h(u+ct))\exp(hS(s))E\left[\exp(h(S(t)-S(s)))\right] \\ = \ & \exp(-h(u+ct))\exp(hS(s))E\left[\exp(h(S(t-s)))\right]. \end{aligned}$$

Further,

$$\begin{aligned} & E\left[\exp(h(S(t-s))\right] \\ = \ & E\left[\exp(h(\sum_{i=1}^{N(t-s)} X_i))\right] \\ = \ & \sum_{m\geq 0} E\left[\mathbf{1}_{\{N(t-s)=m\}}\exp(h(\sum_{i=1}^m X_i))\right] \\ = \ & \sum_{m\geq 0} E\left[\mathbf{1}_{\{N(t-s)=m\}}\right] E\left[\exp(h(\sum_{i=1}^m X_i))\right] \\ = \ & \sum_{m\geq 0} E\left[\mathbf{1}_{\{N(t-s)=m\}}\right] \left(E\left[\exp(hX_1)\right]\right)^m. \end{aligned}$$

On the other hand, if h is the adjustment coefficient, we find that

$$E\left[\exp(hX_1)\right] = \frac{1}{E\left[\exp(-hcW_1)\right]}.$$

Here

$$\begin{aligned} E\left[\exp(-hcW_1)\right] &= \int_0^\infty \exp(-hcy)\lambda \exp(-\lambda y)dy \\ &= \lambda \int_0^\infty \exp(-(hc+\lambda)y)dy \\ &= \lambda/(hc+\lambda), \end{aligned}$$

since $W_1 \sim Exp(\lambda)$. So

$$E\left[\exp(hX_1)\right] = \frac{hc+\lambda}{\lambda}.$$

Thus

$$\begin{aligned} E\left[\exp(h(S(t-s))\right] &= \sum_{m\geq 0} E\left[\mathbf{1}_{\{N(t-s)=m\}}\right] \left(E\left[\exp(hX_1)\right]\right)^m \\ &= \sum_{m\geq 0} E\left[\mathbf{1}_{\{N(t-s)=m\}}\right] (\frac{hc+\lambda}{\lambda})^m \\ &= \sum_{m\geq 0} e^{-\lambda(t-s)} \frac{(\lambda(t-s))^m}{m!} (\frac{hc+\lambda}{\lambda})^m \\ &= e^{-\lambda(t-s)} \exp((t-s)(hc+\lambda)) \\ &= \exp((t-s)hc). \end{aligned}$$

Therefore, we get that

$$\begin{aligned} E\left[\exp(-hU(t))|\ S(s)\right] &= \exp(-h(u+ct)) \exp(hS(s)) \exp((t-s)hc) \\ &= \exp(-h(u+cs)) \exp(hS(s)) \\ &= \exp(-hU(s)), s<t. \end{aligned}$$

(iii) It follows from (ii) for $s=0$ that

$$\begin{aligned} E\left[\exp(-hU(t))|\ S(0)\right] &= \exp(-hU(0)) = \exp(-h(u+c\cdot 0)) \exp(hS(0)) \\ &= \exp(-hu). \end{aligned}$$

So by using the properties of conditional expectations and taking the expectation on both sides of the latter equation, we obtain that

$$E\left[\exp(-hU(t))\right] = E\left[E\left[\exp(-hU(t))|\ S(0)\right]\right] = \exp(-hu), t \geq 0.$$