

Prob. 1 : (i) Calculate the maximum likelihood estimator $\hat{\lambda}$ based on the Danish fire insurance data from 02/09/1987 until 02/22/1987, that is maximize the likelihood function

$$L(\lambda) = \prod_{j=1}^n f_{W_j}(\lambda; W_j) \quad \text{w.r.t. } \lambda > 0$$

with i.i.d inter-arrival times $W_i \sim \text{Exp}(\lambda)$

$$\Rightarrow L(\lambda) = \prod_{j=1}^n \lambda \exp(-\lambda W_j) = \lambda^n \exp(-\lambda (W_1 + \dots + W_n))$$

$\underbrace{W_1 + \dots + W_n}_{= T_n}$

$$\Rightarrow L \text{ attains max. in } \hat{\lambda} = \frac{n}{T_n}$$

data $\rightarrow n=10, T_n=13$

$$\Rightarrow \hat{\lambda} = \frac{10}{13} \approx 0.7692$$

(ii) distr. of T_n

Since $T_n = W_1 + \dots + W_n$ with i.i.d $W_i \sim \text{Exp}(\hat{\lambda})$ we get that

$$P(T_n \leq x) = 1 - e^{-\hat{\lambda}x} \sum_{k=0}^{n-1} \frac{(\hat{\lambda}x)^k}{k!}, \quad x \geq 0 \quad (*)$$

Consider e.g. $n=2$:

$$P(T_2 \leq x) = P(W_1 + W_2 \leq x) = E\left[1_{\{0 \leq W_1 + W_2 \leq x\}}(W_1, W_2)\right] \\ = E[F(W_1, W_2)]$$

where $F(w_1, w_2) = 1_{\{0 \leq w_1 + w_2 \leq x\}}$ and

where (W_1, W_2) has joint density

$$g(w_1, w_2) = \hat{\lambda} e^{-\hat{\lambda} w_1} \cdot \hat{\lambda} e^{-\hat{\lambda} w_2}$$

$$\Rightarrow P(T_2 \leq x) = \int_0^\infty \int_0^\infty F(w_1, w_2) g(w_1, w_2) dw_1 dw_2$$

$$= \int_0^x e^{-\hat{\lambda} w_2} \int_0^{x-w_2} \hat{\lambda}^2 e^{-\hat{\lambda} w_1} dw_1 dw_2$$

$$= 1 - e^{-\hat{\lambda}x} \left(\sum_{k=0}^{n-1} \frac{(\hat{\lambda}x)^k}{k!} \right)$$

$$\Rightarrow (*) \quad f_{T_n}(x) = e^{-\hat{\lambda}x} \hat{\lambda}^n \frac{x^{n-1}}{\frac{\Gamma(n)}{(n-1)!}}, \quad x > 0 \quad \Rightarrow$$

T_n Gamma distr. with shape parameter $\alpha = n$ and scale param. $\lambda = \hat{\lambda}$

$\Rightarrow E[T_n] = \frac{\lambda}{\lambda} = \frac{n}{\lambda} = \frac{13 \cdot n}{10}$ and
 $Var[T_n] = \frac{\lambda}{\lambda^2} = \frac{n}{\lambda^2} = \left(\frac{13}{10}\right)^2 \cdot n$
 $E[W_1] = 1.3$ expected inter-arrival time

Prob. 2 : We have to show that

$$P(B(t) \leq x_1, F(t) \leq x_2) = P(B(t) \leq x_1) \cdot P(F(t) \leq x_2), \quad x_1, x_2 \geq 0$$

Let $x_1 < t$ then

$$\{B(t) \leq x_1\} \stackrel{\text{def. of } B(t)}{=} \{T_{N(t)} \leq t\} \quad \{t - x_1 \leq T_{N(t)} \leq t\}$$

$$= \{N(t - x_1, t] \geq 1\} = \text{at least one jump of } S \text{ on } (t - x_1, t] \quad (+)$$

$$\Rightarrow P(B(t) \leq x_1) = 1 - P(N(t - x_1, t] = 0) = 1 - e^{-\lambda x_1}$$

If $x_1 \geq t$ then

$$P(B(t) \leq x_1) = P(\underbrace{t - x_1}_{\leq 0} \leq T_{N(t)} \leq t) = 1$$

On the other hand

$$\{F(t) \leq x_2\} \stackrel{\text{def. of } F(t)}{=} \{t < T_{N(t)+1} \leq t + x_2\}$$

$$= \{N(t, t + x_2] \geq 1\}, \quad x_2 > 0$$

$$\Rightarrow P(F(t) \leq x_2) = 1 - e^{-\lambda x_2}, \quad x_2 \geq 0 \quad (++)$$

w.l.o.g. let $x_1 < t, x_2 > 0$
Then

$$P(B(t) \leq x_1, F(t) \leq x_2) \stackrel{(+), (++)}{=} P(N(t - x_1, t] \geq 1, N(t, t + x_2] \geq 1)$$

$\stackrel{N \text{ has indep. increm.}}{=} \underbrace{P(N(t - x_1, t] \geq 1)}_{\stackrel{(+)}{=} P(B(t) \leq x_1)} \cdot \underbrace{P(N(t, t + x_2] \geq 1)}_{\stackrel{(++)}{=} P(F(t) \leq x_2)}$

$\rightarrow$ Prob. 2 is called inspection paradox, since intuition says that

$$P(B(t) \leq x_1) < 1 - e^{-\lambda x_1}, \quad x_1 < t$$

and

$$P(F(t) \leq x_2) < 1 - e^{-\lambda x_2}, \quad x_2 > 0$$

Exercises 1

Prob. 4 :

(i) model assumption on the claim size distr. :

$$F \sim \text{Exp}(\hat{\lambda})$$

$\hat{\lambda}$ maximum likelihood estimator based on the claim size data X_i^* :

$$X_1 = 7.733848 \leftarrow X_{(11)}^*$$

$$X_2 = 1.437954 \leftarrow X_{(3)}^*$$

$$X_3 = 1.175275 \leftarrow X_{(1)}^*$$

$$X_4 = 1.501669 \leftarrow X_{(4)}^*$$

$$X_5 = 1.506003 \leftarrow X_{(5)}^*$$

$$X_6 = 5.925473 \leftarrow X_{(9)}^*$$

$$X_7 = 15.692992 \leftarrow X_{(12)}^*$$

$$X_8 = 1.436040 \leftarrow X_{(2)}^*$$

$$X_9 = 3.770965 \leftarrow X_{(8)}^*$$

$$X_{10} = 6.716554 \leftarrow X_{(10)}^*$$

$$X_{11} = 1.104229 \leftarrow X_{(6)}^*$$

$$X_{12} = 2.825732 \leftarrow X_{(7)}^*$$

→ Likelihood function :

$$L(\lambda) = \prod_{i=1}^n (\lambda \cdot e^{-\lambda X_i}) = \lambda^n e^{-\lambda (\sum_{i=1}^n X_i)} \rightarrow \max_{\lambda}$$

$$\Rightarrow \hat{\lambda} = \frac{n}{\sum_{i=1}^n X_i} \quad \stackrel{n=12}{\Rightarrow} \hat{\lambda} = 0.232889$$

QQ-plot against F :

$$\left\{ \left(X_{(k)}^*, F^{\leftarrow} \left(\frac{k}{13} \right) \right), k=1, \dots, 12 \right\}$$

Since

$$F(x) = 1 - e^{-\hat{\lambda}x}, \quad x \geq 0$$

has the inverse

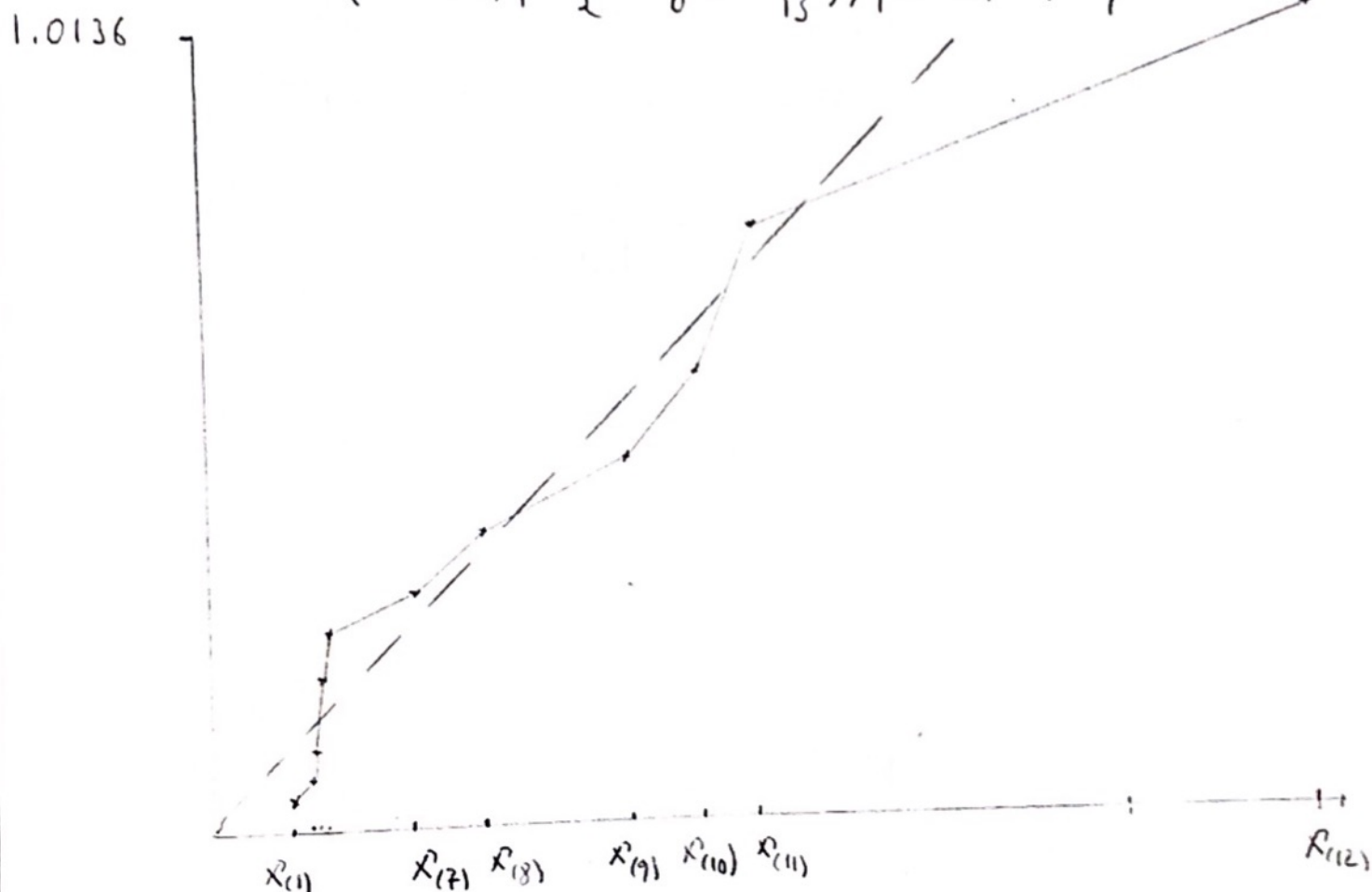
$$F^{-1}(t) = -\frac{1}{\hat{\lambda}} \log(1-t)$$

we have that

$$F^{\leftarrow}(t) = F^{-1}(t)$$

$\Rightarrow$ QQ-plot :

$$\left\{ \left(X_{(k)}, -\frac{1}{\lambda} \log\left(1 - \frac{k}{13}\right) \right), k=1, \dots, 12 \right\}$$



If the data set was large, the graph would indicate a (real) claim size distr. which is heavier tailed than $\text{Exp}(\hat{\lambda})$

Exerc. 1

Prob. 5 : The NPC is given by

$$E[X_1] - C E[W_1] < 0.$$

Using the empirical mean, we obtain that

$$E[X_1] = 4.186 \text{ and } E[W_1] = 3.75$$

So

$$C > \frac{E[X_1]}{E[W_1]} = 1.116$$