

Non-Life Insurance Mathematics (STK4540)

Løsningsforslag prøveeksamen

Problem 1 (i) Since the claim numbers are modelled by a Poisson process, the inter-arrival times $W_i, i \geq 1$ are *i.i.d.* with common distribution $W_1 \sim \text{Exp}(\lambda)$. So by Exercises 1, Problem 1 we know that the MLE $\hat{\lambda}$ is given by

$$\hat{\lambda} = \frac{n}{\sum_{i=1}^n W_i}.$$

The observed arrival times are $W_1 = 1$ (the day 09/30/1990 is excluded), $W_2 = 1, W_3 = 1, W_4 = 1, W_5 = 3, W_6 = 1, W_7 = 2$. So $n = 7$ (sample size) and we get that

$$\hat{\lambda} = \frac{7}{10} = 0.7.$$

(ii) Recall that

$$\Psi(u) \sim \rho^{-1} \bar{F}_{X_1, I}^*(u)$$

for $u \rightarrow \infty$ (Th. 3.4.13).

Just as in the 1. assignment we want to approximate

$$F_{X_1, I}^*(u) = \frac{1}{E^*[X_1]} \int_0^u P^*(X_1 > y) dy$$

by means of the empirical distribution function F_n .

So we see that

$$E^*[X_1] \approx \frac{1}{n} \sum_{i=1}^n X_i \stackrel{n=7}{=} 27.201.$$

On the other hand,

$$P^*(X_1 > y) \approx 1 - F_n(y).$$

Hence,

$$\begin{aligned} \int_0^u P^*(X_1 > y) dy &\approx \int_0^u \left(1 - \frac{1}{n} \sum_{i=1}^n 1_{(-\infty, y]}(X_i)\right) dy \\ &= u - \frac{1}{n} \sum_{i=1}^n (\max(u, X_i) - X_i) \approx 20.821. \end{aligned}$$

Using the latter, we find that

$$\Psi(100) = 78.183\%.$$

Problem 2 (i) We apply Theorem 6.3.1 in Mikosch and obtain that

$$\hat{\mu}_{LB} = (1 - w)\mu + w\bar{X},$$

where

$$w = \frac{n\lambda}{\varphi + n\lambda} \text{ and } \bar{X} = \frac{1}{n} \sum_{j=1}^n X_j.$$

Here:

1.

$$\mu = E[\mu(\theta)] = E[\theta] \stackrel{\text{hint}}{=} \frac{2.9}{2} = 1.45.$$

2.

$$\lambda = \text{Var}[\mu(\theta)] = \text{Var}[\theta] \stackrel{\text{hint}}{=} \frac{2.9}{2^2} = 0.725.$$

3.

$$\varphi = E[\text{Var}[X_1 | \theta]] = E[\theta] \stackrel{\text{hint}}{=} \frac{2.9}{2} = 1.45.$$

Thus

$$w \stackrel{n=10}{=} \frac{10 \cdot 0.725}{1.45 + 10 \cdot 0.725} = 0.833 \text{ and } \bar{X} = 1.5.$$

Therefore, we get that

$$\hat{\mu}_{LB} = (1 - w)\mu + w\bar{X} \approx 1.492.$$

expected claim number (per year) given the observations.

The corresponding risk is given by

$$\rho(\hat{\mu}_{LB}) = (1 - w)\lambda = 0.167 \cdot 0.725 \approx 0.121.$$

(ii) We now assume in (i) that $\theta \sim U(0, 1)$. Then

1.

$$\mu = E[\mu(\theta)] = E[\theta] = \int_0^1 y dy = 0.5.$$

2.

$$\begin{aligned} \lambda &= \text{Var}[\mu(\theta)] = E[(\mu(\theta))^2] - \mu^2 \\ &= E[\theta^2] - \mu^2 \\ &= \int_0^1 y^2 dy - \mu^2 = \frac{1}{3} - \frac{1}{4} \approx 0.083. \end{aligned}$$

3.

$$\varphi = E[\text{Var}[X_1 | \theta]] = E[\theta] = 0.5.$$

So

$$w \stackrel{n=10}{=} \frac{10 \cdot 0.083}{0.5 + 10 \cdot 0.083} \approx 0.624 \text{ and } \bar{X} = 1.5.$$

Hence

$$\begin{aligned} \hat{\mu}_{LB} &= (1 - w)\mu + w\bar{X} \\ &= 0.376 \cdot 0.5 + 0.624 \cdot 1.5 \approx 1.124. \end{aligned}$$

Problem 3 Again by Theorem 6.3.1 in Mikosch we have that

$$\hat{\mu}_{LB} = (1 - w)\mu + w\bar{X},$$

where

$$w = \frac{n\lambda}{\varphi + n\lambda} \text{ and } \bar{Y} = \frac{1}{n} \sum_{j=1}^n Y_j.$$

By Exercises 5, Prob. 2 we obtain that

1.

$$\begin{aligned} \mu &= E[p(\theta)] = E[P(X_1 > K | \theta)] = E[(\frac{\lambda'}{K})^\theta] \\ &= \left(\frac{\beta}{\beta - \log(\lambda'/K)} \right)^\gamma \approx 0.216. \end{aligned}$$

2.

$$\begin{aligned} \lambda &= Var[p(\theta)] = Var[(\frac{\lambda'}{K})^\theta] \\ &= \left(\frac{\beta}{\beta - 2\log(\lambda'/K)} \right)^\gamma - \left(\frac{\beta}{\beta - \log(\lambda'/K)} \right)^{2\gamma} \approx 0.0447. \end{aligned}$$

3.

$$\begin{aligned} \varphi &= E[Var[X_1 | \theta]] = E[p(\theta)] - E[(p(\theta))^2] \\ &= \left(\frac{\beta}{\beta - \log(\lambda'/K)} \right)^\gamma - \left(\frac{\beta}{\beta - 2\log(\lambda'/K)} \right)^\gamma \approx 0.124. \end{aligned}$$

So

$$w \stackrel{n=10}{=} \frac{10 \cdot 0.0447}{0.124 + 10 \cdot 0.0447} \approx 0.783 \text{ and } \bar{Y} = 0.2.$$

Therefore, we get that

$$\hat{\mu}_{LB} = (1 - w)\mu + w\bar{Y} \approx 20.387\%$$

estimated probability for $X_1 > 10000$ NOK given θ .

The corresponding risk is given by

$$\rho(\hat{\mu}_{LB}) = (1 - w)\lambda \approx 0.0097.$$

Problem 4 As before we want to use Theorem 6.3.1 in Mikosch and get that

$$\hat{\mu}_{LB} = (1 - w)\mu + w\bar{X},$$

where

$$w = \frac{n\lambda}{\varphi + n\lambda} \text{ and } \bar{X} = \frac{1}{n} \sum_{j=1}^n X_j.$$

Here:

1.

$$\begin{aligned}
\mu &= E[\mu(\theta)] = E[E[\exp(\theta + \tau Z_1) | \theta]] \\
&= E[\exp(\theta) E[\exp(\tau Z_1) | \theta]] \\
&\stackrel{\theta, Z_1 \text{ independent}}{=} E[\exp(\theta) E[\exp(\tau Z_1)]] \\
&\stackrel{\text{hint in Ex4, Prob.4}}{=} E[\exp(\theta) \exp(\frac{1}{2} \tau^2)] \\
&\stackrel{\text{hint in Ex4, Prob.4}}{=} \exp(\mu' + \frac{1}{2}(\tau^2 + \sigma^2)) \\
&\approx 29.224.
\end{aligned}$$

2.

$$\begin{aligned}
\lambda &= \text{Var}[\mu(\theta)] = E[(\mu(\theta))^2] - \mu^2 \\
&= E[(\mu(\theta))^2] - \mu^2 \\
&= E[(\exp(\theta) \exp(\frac{1}{2} \tau^2))^2] - \mu^2 \\
&= E[\exp(2\theta) \exp(\tau^2)] - \mu^2 \\
&= \exp(2\mu' + \tau^2 + 2\sigma^2) - \mu^2 \\
&\approx 554.063.
\end{aligned}$$

3.

$$\begin{aligned}
\varphi &= E[\text{Var}[X_1 | \theta]] = E[E[X_1^2 | \theta]] - E[(E[X_1 | \theta])^2] \\
&= E[E[\exp(2\theta + 2\tau Z_1) | \theta]] - E[\exp(2\theta) \exp(\tau^2)] \\
&= E[\exp(2\theta) E[\exp(2\tau Z_1)]] - E[\exp(2\theta) \exp(\tau^2)] \\
&= E[\exp(2\theta) \exp(2\tau^2)] - E[\exp(2\theta) \exp(\tau^2)] \\
&= \exp(2\mu' + 2\sigma^2 + 2\tau^2) - \exp(2\mu' + 2\sigma^2 + \tau^2) \\
&\approx 399.937.
\end{aligned}$$

So

$$w = \frac{8 \cdot 554.063}{399.937 + 8 \cdot 554.063} \approx 0.917$$

and

$$\begin{aligned}
\hat{\mu}_{LB} &= (1 - w)\mu + w\bar{X} \\
&= 0.083 \cdot 29.224 + 0.917 \cdot 15.75 \approx 16.868 \text{ (in 1000 NOK)}.
\end{aligned}$$

Problem 5 (i) We want to calculate

$$\hat{\mu}_B = E[E[X_1 | \theta] | X_1, \dots, X_n].$$

To this end, we shall compute the conditional density $f_\theta(y | X_1 = x_1, \dots, X_n = x_n)$ by means of L. 5.2.3 (2) in T. Mikosch:

$$\begin{aligned}
&f_\theta(y | X_1 = x_1, \dots, X_n = x_n) \\
&= \frac{f_\theta(y) f_{X_1}(x_1 | \theta = y) \dots f_{X_n}(x_n | \theta = y)}{f_{X_1, \dots, X_n}(x_1, \dots, x_n)}.
\end{aligned}$$

1. $f_\theta(y)$:

$$f_\theta(y) = \frac{\beta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\beta y}, y > 0.$$

2. $f_{X_1}(x | \theta = y)$:

$$f_{X_1}(x | \theta = y) = y e^{-y \cdot x}.$$

3. $f_{X_1, \dots, X_n}(x_1, \dots, x_n)$:

$$\begin{aligned} & F_{X_1, \dots, X_n}(x_1, \dots, x_n) \\ &= P(X_1 \leq x_1, \dots, X_n \leq x_n) = E[P(X_1 \leq x_1, \dots, X_n \leq x_n | \theta)] \\ & \stackrel{X_i, i \geq 1 \text{ i.i.d. given } \theta}{=} E\left[\prod_{j=1}^n P(X_1 \leq x_j | \theta)\right] = E\left[\prod_{j=1}^n (1 - e^{-\theta x_j})\right]. \end{aligned}$$

So differentiation w.r.t. $x_1, \dots, x_n$ of $F_{X_1, \dots, X_n}(x_1, \dots, x_n)$ combined with dominated convergence gives

$$\begin{aligned} & f_{X_1, \dots, X_n}(x_1, \dots, x_n) \\ &= \frac{\partial^n}{\partial x_1 \dots \partial x_n} F_{X_1, \dots, X_n}(x_1, \dots, x_n) \\ &= E\left[\prod_{j=1}^n \theta e^{-\theta x_j}\right] = E[\theta^n e^{-\theta(\sum_{j=1}^n x_j)}] \\ &= \int_0^\infty y^n e^{-y(\sum_{j=1}^n x_j)} \frac{\beta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\beta y} dy \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty y^{(\alpha+n)-1} e^{-y(\beta + \sum_{j=1}^n x_j)} \frac{(\beta + \sum_{j=1}^n x_j)^{\alpha+n}}{\Gamma(\alpha+n)} dy \frac{\Gamma(\alpha+n)}{(\beta + \sum_{j=1}^n x_j)^{\alpha+n}} \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot 1 \cdot \frac{\Gamma(\alpha+n)}{(\beta + \sum_{j=1}^n x_j)^{\alpha+n}}. \end{aligned}$$

Hence,

$$\begin{aligned} & f_\theta(y | X_1 = x_1, \dots, X_n = x_n) \\ &= \frac{\frac{\beta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\beta y} y^n e^{-y(\sum_{j=1}^n x_j)}}{\frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+n)}{(\beta + \sum_{j=1}^n x_j)^{\alpha+n}}} \\ &= f_{\Gamma(\gamma', \beta')}(y) \text{ (Gamma density)} \end{aligned}$$

with $\gamma' := \alpha + n, \beta' := \beta + \sum_{j=1}^n x_j$.

Therefore

$$\begin{aligned}
& E[E[X_1 | \theta] | X_1 = x_1, \dots, X_n = x_n] \\
&= E\left[\frac{1}{\theta} | X_1 = x_1, \dots, X_n = x_n\right] \\
&= \int_0^\infty \frac{1}{y} \cdot f_{\Gamma(\gamma', \beta')}(y) dy \\
&= \frac{(\beta')^{\gamma'}}{\Gamma(\gamma')} \int_0^\infty y^{(\gamma'-1)-1} e^{-y \cdot \beta'} \frac{(\beta')^{\gamma'-1}}{\Gamma(\gamma'-1)} dy \frac{\Gamma(\gamma'-1)}{(\beta')^{\gamma'-1}} \\
&= \frac{(\beta')^{\gamma'}}{\Gamma(\gamma')} \frac{\Gamma(\gamma'-1)}{(\beta')^{\gamma'-1}} = \frac{\beta' \Gamma(\gamma'-1)}{\Gamma(\gamma')} \\
&\quad \Gamma(z+1) \stackrel{=}{=} z \Gamma(z) \quad \frac{\beta'}{(\gamma'-1)} \\
&= \frac{\beta + \sum_{j=1}^n x_j}{\alpha + n - 1}.
\end{aligned}$$

So

$$\hat{\mu}_B = \frac{\beta + \sum_{j=1}^n x_j}{\alpha + n - 1} = \frac{1 + 71}{3 + 10 - 1} = 6000 \text{ NOK}.$$

Corresponding risk $\rho(\hat{\mu}_L)$:

$$\rho(\hat{\mu}_L) = E[\text{Var}[\mu(\theta) | X_1, \dots, X_n]].$$

On the other hand,

$$\begin{aligned}
& \text{Var}[\mu(\theta) | X_1, \dots, X_n] \\
&= E[(\mu(\theta))^2 | X_1, \dots, X_n] - (E[\mu(\theta) | X_1, \dots, X_n])^2.
\end{aligned}$$

So

$$\begin{aligned}
\rho(\hat{\mu}_L) &= E[(\mu(\theta))^2] - E[(E[\mu(\theta) | X_1, \dots, X_n])^2] \\
&= E\left[\frac{1}{\theta^2}\right] - E\left[\left(\frac{\beta + \sum_{j=1}^n X_j}{\alpha + n - 1}\right)^2\right].
\end{aligned}$$

1.

$$\begin{aligned}
& E\left[\left(\frac{\beta + \sum_{j=1}^n X_j}{\alpha + n - 1}\right)^2\right] \\
&= \frac{1}{(\alpha + n - 1)^2} (\beta^2 + 2\beta \sum_{j=1}^n E[X_j] + \sum_{i,j=1}^n E[X_i X_j]) \\
&= \frac{1}{(\alpha + n - 1)^2} (\beta^2 + 2\beta n E\left[\frac{1}{\theta}\right] + \sum_{j=1}^n E[E[X_j^2 | \theta]] + \sum_{\substack{i,j=1 \\ i \neq j}}^n E[E[X_i X_j | \theta]]) \\
&= \frac{1}{(\alpha + n - 1)^2} (\beta^2 + 2\beta n E\left[\frac{1}{\theta}\right] + n E[E[X_1^2 | \theta]] + \sum_{\substack{i,j=1 \\ i \neq j}}^n E[E[X_i | \theta] E[X_j | \theta]]) \\
&= \frac{1}{(\alpha + n - 1)^2} (\beta^2 + 2\beta n E\left[\frac{1}{\theta}\right] + n E[E[X_1^2 | \theta]] + (n^2 - n) E\left[\frac{1}{\theta^2}\right])
\end{aligned}$$

We have that

$$\begin{aligned}
& E[X_1^2 | \theta = y] \\
&= \int_0^\infty z^2 y e^{-yz} dz \\
&= y \int_0^\infty z^{3-1} e^{-yz} \frac{y^3}{\Gamma(3)} dz \frac{\Gamma(3)}{y^3} \\
&= y \cdot 1 \cdot \frac{\Gamma(3)}{y^3} = \frac{\Gamma(3)}{y^2} = \frac{2}{y^2}.
\end{aligned}$$

Thus

$$E[E[X_1^2 | \theta]] = 2E\left[\frac{1}{\theta^2}\right].$$

Further,

$$\begin{aligned}
E\left[\frac{1}{\theta^2}\right] &= \int_0^\infty y^{-2} \frac{\beta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\beta y} dy \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty y^{(\alpha-2)-1} e^{-\beta y} \frac{\beta^{\alpha-2}}{\Gamma(\alpha-2)} dy \frac{\Gamma(\alpha-2)}{\beta^{\alpha-2}} \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha-2)}{\beta^{\alpha-2}} = \frac{\beta^2 \Gamma(\alpha-2)}{\Gamma(\alpha)} \\
&\stackrel{\Gamma(z+1)=z\Gamma(z)}{=} \frac{\beta^2 \Gamma(\alpha-2)}{(\alpha-1)\Gamma(\alpha-1)} \\
&\stackrel{\Gamma(z+1)=z\Gamma(z)}{=} \frac{\beta^2 \Gamma(\alpha-2)}{(\alpha-1)(\alpha-2)\Gamma(\alpha-2)} \\
&= \frac{\beta^2}{(\alpha-1)(\alpha-2)}.
\end{aligned}$$

Similarly, we see that

$$\begin{aligned}
E\left[\frac{1}{\theta}\right] &= \int_0^\infty y^{-1} \frac{\beta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\beta y} dy \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty y^{(\alpha-1)-1} e^{-\beta y} \frac{\beta^{\alpha-1}}{\Gamma(\alpha-1)} dy \frac{\Gamma(\alpha-1)}{\beta^{\alpha-1}} \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha-1)}{\beta^{\alpha-1}} = \frac{\beta}{\alpha-1}.
\end{aligned}$$

So

$$\begin{aligned}
& E\left[\left(\frac{\beta + \sum_{j=1}^n X_j}{\alpha + n - 1}\right)^2\right] \\
&= \frac{1}{(\alpha + n - 1)^2} \left(\beta^2 + 2\beta n \frac{\beta}{\alpha - 1} + n \frac{2\beta^2}{(\alpha - 1)(\alpha - 2)} + (n^2 - n) \frac{\beta^2}{(\alpha - 1)(\alpha - 2)} \right) \\
&= \frac{1}{(\alpha + n - 1)^2} \left(\beta^2 + 2\beta n \frac{\beta}{\alpha - 1} + n \frac{\beta^2}{(\alpha - 1)(\alpha - 2)} + n^2 \frac{\beta^2}{(\alpha - 1)(\alpha - 2)} \right).
\end{aligned}$$

Altogether, we obtain that

$$\begin{aligned}
& \rho(\hat{\mu}_L) \\
= & E\left[\frac{1}{\theta^2}\right] - E\left[\left(\frac{\beta + \sum_{j=1}^n X_j}{\alpha + n - 1}\right)^2\right] \\
= & \frac{\beta^2}{(\alpha - 1)(\alpha - 2)} \\
& - \frac{1}{(\alpha + n - 1)^2}(\beta^2 + 2\beta n \frac{\beta}{\alpha - 1} + n \frac{\beta^2}{(\alpha - 1)(\alpha - 2)} + n^2 \frac{\beta^2}{(\alpha - 1)(\alpha - 2)}) \\
\approx & 0.0417.
\end{aligned}$$

(ii) Using once again Theorem 6.3.1 in Mikosch we get that

$$\hat{\mu}_{LB} = (1 - w)\mu + w\bar{X},$$

where

$$w = \frac{n\lambda}{\varphi + n\lambda} \text{ and } \bar{X} = \frac{1}{n} \sum_{j=1}^n X_j.$$

1.

$$\mu = E[\mu(\theta)] = E\left[\frac{1}{\theta}\right] \stackrel{(i)}{=} \frac{\beta}{\alpha - 1} = \frac{1}{2}.$$

$$\begin{aligned}
\lambda &= Var[\mu(\theta)] = E[(\mu(\theta))^2] - (E[\mu(\theta)])^2 \\
&= E\left[\frac{1}{\theta^2}\right] - (E\left[\frac{1}{\theta}\right])^2 \\
&\stackrel{(i)}{=} \frac{\beta^2}{(\alpha - 1)(\alpha - 2)} - \left(\frac{\beta}{\alpha - 1}\right)^2 \\
&= \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.
\end{aligned}$$

3.

$$\begin{aligned}
\varphi &= E[Var[X_1 | \theta]] = E[E[X_1^2 | \theta]] - E[(\mu(\theta))^2] \\
&\stackrel{(i)}{=} 2E\left[\frac{1}{\theta^2}\right] - E\left[\frac{1}{\theta^2}\right] \\
&= \frac{\beta^2}{(\alpha - 1)(\alpha - 2)} = \frac{1}{2}.
\end{aligned}$$

So

$$w \stackrel{n=10}{=} \frac{10 \cdot 0.25}{0.5 + 10 \cdot 0.25} \approx 0.833 \text{ and } \bar{X} = 7.1.$$

Altogether, we get that

$$\hat{\mu}_{LB} = (1 - w)\mu + w\bar{X} \approx 5998 \text{ NOK}.$$

Problem 6 See Problem 1, Exercises 7 for the solution.