

⑪ (iii) $\lim_{i \rightarrow \infty} P(A_i) = P(A)$, if $A_1 \subset A_2 \subset \dots$ and

$$A = \bigcup_{i=1}^{\infty} A_i$$

(e.g. $A_i = "S_t \geq \frac{100}{i} \text{ \$/} ", i \in \mathbb{N})$

Examples of prob. meas.

a) $\Omega = \{1, 2, \dots, 6\}$, $\mathcal{F} = \mathcal{B}(\Omega)$ class of all subsets of Ω

$$P(A) \stackrel{\text{def}}{=} (\text{number of elements in } A) / 6$$

e.g. $P(\{1, 2, 3\}) = \frac{1}{2} \rightarrow$ prob. that one throws 1, 2 or 3

b) Dirac measure δ_w :

$$\delta_w(A) \stackrel{\text{def}}{=} \begin{cases} 1, & w \in A \\ 0, & w \notin A \end{cases}$$

c) Lebesgue (-Borel) measure on $[0, 1]$

$$\Omega = [0, 1], \quad \mathcal{F} = \mathcal{B}(\mathbb{R}) \cap [0, 1] = \{A \cap [0, 1] : A \in \mathcal{B}(\mathbb{R})\}$$

One shows that there exists a unique prob. meas.

$$\lambda : \mathcal{F} \rightarrow [0, 1] \text{ s.t.}$$

$$\lambda([a, b]) = b - a$$

$\uparrow$ (length of the interval $[a, b]$)

for all $0 \leq a \leq b \leq 1$.

Ex.: S uniformly distributed on $[0, 1]$

$$\iff P(a \leq S \leq b) = b - a = \lambda([a, b])$$

$$\Rightarrow \text{distr of } S = \text{Lebesgue meas. on } [0, 1]$$

Def. 2.4 (Random variable)

A function

$$X : \Omega \rightarrow \mathbb{R}$$

is called random variable (r.v.) on $(\Omega, \mathcal{F}, P)$, if the set

$$A := \{\omega \in \Omega : X(\omega) \leq t\}$$

is an event for all $t \in \mathbb{R}$, i.e.

for all $t \in \mathbb{R}$ $A \in \mathcal{F}$ (collection of all events)

⑫ Ex. $\mathcal{F}$ entirety of market information up to time T
 $\longrightarrow$ natural to assume that
 $\{S_T \leq t\} \longleftarrow$ stock price at time T

is a (possible) market event for all t ($\$$), i.e.

$$\{S_T \leq t\} \in \mathcal{F} \text{ for all } t$$

$\Rightarrow S_T$ random variable

Def. 2.5 (Random vector)

A function $X: \Omega \rightarrow \mathbb{R}^n$

with $X(\omega) = (X_1(\omega), \dots, X_n(\omega))$

is called random vector, if $X_1, \dots, X_n$ are r.v.'s

Def. 2.6 (Stochastic process)

Let $\mathcal{I} \neq \emptyset$ be an index set (e.g. $\mathcal{I} = [0, 1]$)

A collection

$$\{X_i\}_{i \in \mathcal{I}}$$

of random variables is called stochastic process.

$\mathcal{I}$ is the parameter space.

Ex. $\mathcal{I} = [0, \infty)$, S_t price of a stock at time t .

$\longrightarrow \{S_t\}_{0 \leq t < \infty}$ stoch. proc.

Def. 2.7 (P-null sets)

An event

$$A \in \mathcal{F}$$

s.t.

$$P(A) = 0$$

is called P-null set

Ex. $A = \{S_t = \infty\}$ should be a P-null set, i.e. $P(A) = 0$

Def. 2.8 (P-almost everywhere property)

We say a property Π holds P-almost everywhere (P-a.e.) if there ex. a null set $N \in \mathcal{F}$ s.t.

Π holds for all $\omega \in N^c = \Omega \setminus N$

⑬ Ex.: X_i : i -th fire loss claim, $i=1,2$

$$\Pi = "X_1 \leq X_2"$$

$\rightarrow \Pi$ holds P -a.e. $\iff$

$$X_1(\omega) \leq X_2(\omega)$$

for all $\omega \in N^c = \Omega \setminus N$ for some null set $N \in \hat{\mathcal{F}}$.

Def. 2.9 (Integral w.r.t. P)

Let $X \geq 0$ be a non-negative r.v. Then

$$E[X] \stackrel{\text{def}}{=} \int_{\Omega} X(\omega) P(d\omega) \leftarrow \text{notation}$$

$$\stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} \sum_{i=1}^{n \cdot 2^n} i \cdot 2^{-n} P(A_{i,n})$$

where

$$A_{i,n} := \begin{cases} \{\omega \in \Omega : (i+1)2^{-n} > X(\omega) \geq i \cdot 2^{-n}\}, & i=0, \dots, n \cdot 2^n - 1 \\ \{\omega \in \Omega : X(\omega) \geq n\}, & i=n \cdot 2^n \end{cases}$$

For general X we def.

$$E[X] := \int_{\Omega} X(\omega) P(d\omega) :=$$

$$E[X^+] - E[X^-] \quad (x^+ = \max(x, 0), x^- = \max(-x, 0)),$$

provided $E[X^+], E[X^-] < \infty$ (*)

If (*) holds, X is called (P-) integrable.

Ex.: $\Omega = [0,1]$, $P=2$ (uniform distribution)

$$X(\omega) = \omega^2$$

$$\Rightarrow E[X] = \int_{\Omega} X(\omega) P(d\omega) = \int_{[0,1]} \omega^2 2(d\omega)$$

$$= \int_0^1 \omega^2 d\omega = \frac{1}{3} \omega^3 \Big|_{\omega=0}^1 = \frac{1}{3}$$

Properties:

(i) $E[\alpha X + \beta Y] = \alpha E[X] + \beta E[Y]$, $\alpha, \beta \in \mathbb{R}$ (linearity)

(ii) If $X \leq Y$ P -a.e. then

$$E[X] \leq E[Y]$$

(14) E_X : $X \in \mathbb{N}_0$ r.v.

$$\Rightarrow X = \sum_{n \geq 0} n \cdot \underbrace{\mathbb{1}_{\{X=n\}}}_{=Y_n \text{ r.v.'s}} = \sum_{n \geq 0} n \cdot Y_n$$

$$\mathbb{1}_A(\omega) := \begin{cases} 1, & \omega \in A \\ 0 & \text{else} \end{cases}$$

Using (i) we obtain that

$$\begin{aligned} E[X] &= E\left[\sum_{n \geq 0} n \cdot Y_n\right] = \sum_{n \geq 0} n \underbrace{E[Y_n]}_{=P(\{X=n\})} \\ &= \sum_{n \geq 0} n \cdot P\{X=n\} \end{aligned}$$

Fundamental concept of prob. th $\rightarrow$

Conditional expectation $E[X|A]$ of a r.v. X
w.r.t. a $\mathcal{B}$ -algebra $\mathcal{A} (\subseteq \mathcal{F})$

Heuristically:

$$E[X|\mathcal{A}] = "E[X|A], A \in \mathcal{A}"$$

where $E[X|A] \stackrel{\text{def.}}{=} \frac{1}{P(A)} E[\mathbb{1}_A \cdot X]$ cond. expectation

of X given the event A

Interpretation : $\mathcal{A}$ entirety of market inf. up to time $t \leq T$
maturity
 $\rightarrow \mathcal{A} \subseteq \mathcal{F}$ market inf. up to time T .

S_T stock price at time T

$\rightarrow E[S_T|\mathcal{A}]$ expected terminal stock price
given the entirety of market inf. up to time
 $t \leq T$.

Or more mathematically:

Def 2.10 (cond. expectation w.r.t $\mathcal{A}$) : $\leftarrow$ in general difficult to calculate

Let X be a r.v. s.t. $E[|X|] < \infty$

Then the expected value of X given $\mathcal{A}$ is
the

⑤ unique r.v. Y s.t.

$$E[\mathbb{1}_A \cdot X] = E[\mathbb{1}_A \cdot Y]$$

for all $A \in \mathcal{A}$

and

$\{Y \leq q\} \in \mathcal{A}$ for all $q \in \mathbb{R}$.

The r.v. Y is denoted by $E[X|\mathcal{A}]$

→ Properties of $E[X|\mathcal{A}]$:

(i) $E[\alpha X + \beta Y | \mathcal{A}] = \alpha E[X|\mathcal{A}] + \beta E[Y|\mathcal{A}]$ (linearity)

(ii) $E[E[X|\mathcal{A}]] = E[X]$

(iii) $E[X|\mathcal{A}] = X$, if X r.v. on $(\Omega, \mathcal{A}, P)$

(iv) $E[X|\mathcal{A}] = E[X]$, if X is independent of $\mathcal{A}$, i.e.
 $P(\{X \leq t\} \cap A) = P(X \leq t) \cdot P(A)$ for all $t \in \mathbb{R}, A \in \mathcal{A}$

(v) $E[X|\mathcal{A}] = E[E[X|\mathcal{B}] | \mathcal{A}]$ if $\mathcal{A} \subseteq \mathcal{B}$ σ -algebra

⑥ 3. Ruin theory

→ objective: study of ruin probabilities, i.e. of probabilities for ruin in an insurance portfolio in the case of both small claims and large claims

For this purpose, we have to introduce some basic notions of ruin theory:

3.1 Risk process, Ruin probability, Net Profit Condition

Def. 3.1.1 (Renewal model for $S(t)$)

The total claim amount $S(t)$ in the renewal model is defined by

$$S(t) = \sum_{i=1}^{N(t)} X_i \quad (3.1.1)$$

where

(i) $X_i, i \geq 1$ are i.i.d non-negative r.v.'s. Here X_i models the i -th claim size (e.g. wind storm loss).

(ii) $N(t), t \geq 0$ is the renewal (counting) process, which counts the number of claims arriving by time t and which is defined as

$$N(t) = \# \{ i \geq 1 : T_i \leq t \}, \quad t \geq 0$$

where

$$T_0 := 0, \quad T_n = W_1 + \dots + W_n$$

for an i.i.d sequence $W_i, i \geq 1$ of (P-a.e.) positive r.v.'s.

The r.v. T_i is called the i -th arrival time (of claim X_i), whereas $W_i = T_i - T_{i-1}$ is referred to as the i -th inter-arrival time

and

(iii) $W_i, i \geq 1$ independent of $X_i, i \geq 1$
Recall: $Y_1, \dots, Y_n$ r.v.'s indep. $\iff$
 $P(Y_1 \leq x_1, \dots, Y_n \leq x_n) = P(Y_1 \leq x_1) \dots P(Y_n \leq x_n)$
for all $x_1, \dots, x_n \in \mathbb{R}$.

⑦

Rem 3.1.2 (Poisson proc. as an example of a renewal proc.)

Consider a homogeneous Poisson process, that is a stoch. proc. $N(t), t \geq 0$ on $(\mathbb{R}, \mathcal{F}, P)$ s.t.

(i) $N(0) = 0$ P-a.e.

(ii) N has independent increments:

$$N(t_1) - N(t_0), \dots, N(t_n) - N(t_{n-1})$$

are indep. r.v.'s for all $0 = t_0 < t_1 < \dots < t_n, n \in \mathbb{N}$.

(iii) N has stationary Poisson distributed increments:

$$N(t+h) - N(t) \stackrel{d}{=} N(h) \text{ with } N(h) \sim \text{Pois}(\lambda h)$$

for all $t \geq 0, h > 0$ for some $\lambda > 0$ (jump intensity)

Recall:

$$1. \quad X, Y \text{ r.v.'s} \Rightarrow X \stackrel{d}{=} Y \iff$$

$$P(X \leq x) = P(Y \leq x) \text{ for all } x$$

$$2. \quad M \sim \text{Pois}(\mu) \iff$$

$$P(M=k) = e^{-\mu} \frac{\mu^k}{k!}, k \in \mathbb{N}_0$$

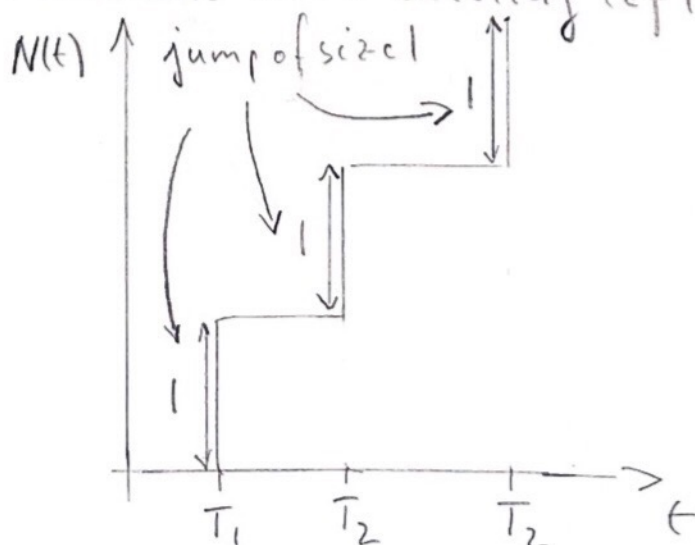
(iv) With probability 1 the sample paths of

$N(t, \omega), t \geq 0$ are cadlag, that is

$\exists \tilde{\Omega} \in \mathcal{F}$ with $P(\tilde{\Omega}) = 1$ s.t. for all $\omega \in \tilde{\Omega}$:

$$(\epsilon \mapsto N(t, \omega))$$

right-continuous with existing left limits



⑮ One shows that

- (i) If N is a hom. Poisson proc. with intensity λ and arrival times $0 \leq T_1 \leq T_2 \leq \dots$ then N is a renewal proc. with i.i.d $W_i \sim \text{Exp}(\lambda), i \geq 1$, that is

$$P(W_i \leq x) = \int_0^x f(y) dy \quad (3.1.2)$$

with probability density

$$f(y) = \lambda e^{-\lambda y}, y \geq 0$$

- (ii) Let N be a renewal process with i.i.d $W_i \sim \text{Exp}(\lambda)$. Then N is a hom. Poisson proc. with intensity $\lambda > 0$.

Ex. 3.1.3 (Cramér-Lundberg-model as an example of a renewal model)

If $N(t), t \geq 0$ is a hom. Poiss. proc. (Def. 3.1.1), then $S(t), t \geq 0$ is the total claim amount in the Cramér-Lundberg-model.

Def. 3.1.4 (Risk process)

The risk process (or surplus) of the portfolio is defined as

$$U(t) = u + p(t) - S(t), t \geq 0,$$

where

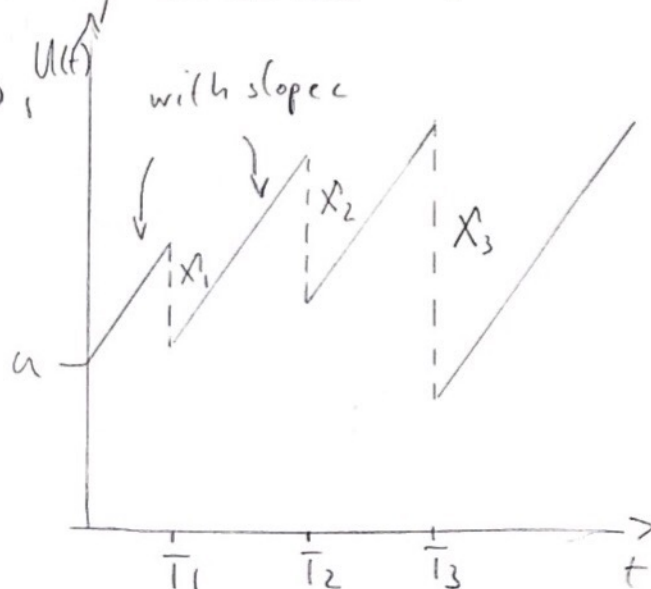
$u > 0$ initial capital (sufficiently large to avoid bankruptcy) ! e.g. reinforced by supervisory authorities

$p(t)$ premium income

Assumption: $p(t) = c \cdot t, t \geq 0$,

where

$c > 0$ premium rate



①9) Def. 3.1.5 (Ruin, ruin time, ruin probability)

(i) The event

$\text{Ruin} := \{U(t) < 0 \text{ for some } t \geq 0\}$
is called ruin

→ event that U ever falls below 0

(ii) The r.v.

$T := \inf \{t \geq 0 : U(t) < 0\} \in [0, \infty]$ ($\inf \emptyset \stackrel{\text{def}}{=} \infty$)
is called ruin time

→ time when ruin occurs

(iii) The probability of ruin is def. as

$$\psi(u) := \underbrace{P(\text{Ruin} | U(0) = u)}_{= P(\text{Ruin})} = P(T < \infty)$$

Note that

$$\begin{aligned} \text{Ruin} &= \bigcup_{t \geq 0} \{U(t) < 0\} = \left\{ \inf_{t \geq 0} U(t) < 0 \right\} \\ &= \{T < \infty\} \end{aligned}$$

→ main objective of insurance business is to keep $\psi(u)$ small for sufficiently large initial capital u

worst case →

Th. 3.1.6 (Ruin with prob. 1)

Let $E[W_1], E[X_1] < \infty$ and

$$E[X_1] - c \cdot E[W_1] \geq 0, \quad (3.1.3)$$

Then

$$P(\text{Ruin} | U(0) = u) = 1$$

for all $u > 0$.

Ex. 3.1.7 : Recall the premium $P(t)$ based on the equivalence principle, that is

$$P(t) = P_{\text{Net}}(t) = E[S(t)]$$

$N(t)$ indep. of the i.i.d. $X_i, i \geq 1$ (Def. 3.1.1(ii))

$$\Rightarrow E[S(t)] = E[N(t)] \cdot E[X_1] \quad (\text{see exerc.})$$

(20) Suppose $N(t)$ hom. Poisson proc. with intensity $\lambda > 0$,
 then $E[S(t)] = \overbrace{\lambda \cdot t}^{= E[N(t)]} E[X_1] = \frac{E[X_1]}{E[W_1]} \cdot t$,

since $E[W_1] = \frac{1}{\lambda}$ (see 3.1.2)

$$\int_0^{\infty} y \cdot \lambda e^{-\lambda y} dy$$

$$\Rightarrow c = \frac{E[X_1]}{E[W_1]} \Rightarrow (3.1.3) = 0 \Rightarrow$$

p_{net} leads to ruin for all $u > 0$

$\Rightarrow$ p_{net} unwise calculation principle