Non-Life Insurance Mathematics (STK4540)
Solutions to the exam

Problem 1 By Theorem 6.3.1 in Mikosch we have that

$$\widehat{\mu}_{LB} = (1-w)\mu + w\overline{Y},$$

where

$$w = \frac{n\lambda}{\varphi + n\lambda} \text{ and } \overline{Y} = \frac{1}{n}\sum_{j=1}^{n} Y_j.$$

By Exercises 5, Prob. 2 we know that
1.

$$\begin{aligned} \mu &= E[p(\theta)] = E[P(X_1 > K\,|\theta)] = E[(\frac{\lambda'}{K})^{\theta}] \\ &= \left(\frac{\beta}{\beta - \log(\lambda'/K)}\right)^{\gamma} \approx 0.120713. \end{aligned}$$

2.

$$\begin{aligned} \lambda &= Var[p(\theta)] = Var[(\frac{\lambda'}{K})^{\theta}] \\ &= \left(\frac{\beta}{\beta - 2\log(\lambda'/K)}\right)^{\gamma} - \left(\frac{\beta}{\beta - \log(\lambda'/K)}\right)^{2\gamma} \approx 0.0244657. \end{aligned}$$

3.

$$\begin{aligned} \varphi &= E[Var[X_1\,|\theta]] = E[p(\theta)] - E[(p(\theta))^2] \\ &= \left(\frac{\beta}{\beta - \log(\lambda'/K)}\right)^{\gamma} - \left(\frac{\beta}{\beta - 2\log(\lambda'/K)}\right)^{\gamma} \approx 0.0816754. \end{aligned}$$

So

$$w \overset{n=10}{=} \frac{10 \cdot 0.0244657}{0.0816754 + 10 \cdot 0.0244657} \approx 0.749717 \text{ and } \overline{Y} = 0.2.$$

Therefore, we get that

$$\widehat{\mu}_{LB} = (1-w)\mu + w\overline{Y} \approx 18.016\%$$

is the estimated probability for $X_1 > 50000$ NOK given θ.

The corresponding risk is given by

$$\rho(\widehat{\mu}_{LB}) = (1-w)\lambda \approx 0.00612335.$$

Problem 2 (i) We apply Theorem 6.3.1 in Mikosch and get that

$$\widehat{\mu}_{LB} = (1-w)\mu + w\overline{X},$$

where

$$w = \frac{n\lambda}{\varphi + n\lambda} \text{ and } \overline{X} = \frac{1}{n}\sum_{j=1}^{n} X_j.$$

Here:

1.

$$\mu = E[\mu(\theta)] = E[\theta] \overset{\text{hint}}{=} \frac{1.9}{1.2} \approx 1.58333.$$

2.

$$\lambda = Var[\mu(\theta)] = Var[\theta] \overset{\text{hint}}{=} \frac{1.9}{(1.2)^2} \approx 1.31944.$$

3.

$$\varphi = E[Var[X_1\,|\theta]] = E[\theta] \overset{\text{hint}}{=} \frac{1.9}{1.2} \approx 1.58333.$$

Thus

$$w \overset{n=10}{=} \frac{10 \cdot 1.31944}{1.58333 + 10 \cdot 1.31944} \approx 0.892857 \text{ and } \overline{X} = 1.4.$$

Therefore, we get that

$$\widehat{\mu}_{LB} = (1-w)\mu + w\overline{X} \approx 1.41964.$$

expected claim number (per year) given the observations.

The corresponding risk is given by

$$\rho(\widehat{\mu}_{LB}) = (1-w)\lambda \approx 0.141369.$$

(ii) We now assume in (i) that $\theta = Z^2$ for $Z \sim Exp(1)$. Then

1.

$$\begin{aligned} \mu &= E[\mu(\theta)] = E[Z^2] = \int_0^\infty y^2 \exp(-y)dy \\ &= 2\int_0^\infty \frac{1}{2} y^{3-1} \exp(-y)dy = 2. \end{aligned}$$

$$\begin{aligned} \lambda &= Var[\mu(\theta)] = E[(\mu(\theta))^2] - \mu^2 \\ &= E[Z^4] - \mu^2 \\ &= \int_0^\infty y^4 \exp(-y)dy - \mu^2 \\ &= 4!\int_0^\infty \frac{1}{4!} y^{5-1} \exp(-y)dy - \mu^2 \\ &= 24 - \mu^2 = 20. \end{aligned}$$

3.

$$\varphi = E[Var[X_1\,|\theta]] = E[\theta] = 2.$$

So

$$w \overset{n=10}{=} \frac{10 \cdot 20}{2 + 10 \cdot 20} \approx 0.99009 \text{ and } \overline{X} = 1.4.$$

Hence

$$\widehat{\mu}_{LB} = (1-w)\mu + w\overline{X} = 1.40594.$$

Problem 3 (i) Since the claim numbers are modelled by a Poisson process, the inter-arrival times $W_i, i \geq 1$ are *i.i.d.* with common distribution $W_1 \sim Exp(\lambda)$. So by Exercises 1, Problem 1 we know that the MLE $\widehat{\lambda}$ is given by

$$\widehat{\lambda} = \frac{n}{\sum_{i=1}^{n} W_i}.$$

The observed arrival times are $W_1 = 9$ (the day 07/30/1988 is excluded), $W_2 = 1, W_3 = 1, W_4 = 2, W_5 = 3, W_6 = 3, W_7 = 3, W_8 = 1, W_9 = 2$. So $n = 9$ (sample size) and we get that

$$\widehat{\lambda} = \frac{9}{25} \approx 0.36.$$

(ii) We know that

$$\Psi(u) \sim \rho^{-1} \overline{F}^*_{X_1,I}(u)$$

for $u \longrightarrow \infty$ (Th. 3.4.13).

Next we want to approximate

$$F^*_{X_1,I}(u) = \frac{1}{E^*[X_1]} \int_0^u P^*(X_1 > y)dy$$

by means of the empirical distribution function F_n.

So we see that

$$E^*[X_1] \approx \frac{1}{n} \sum_{i=1}^{n} X_i \overset{n=9}{=} 7.20921.$$

On the other hand,

$$P^*(X_1 > y) \approx 1 - F_n(y).$$

Hence,

$$\begin{aligned} \int_0^u P^*(X_1 \quad > \quad y)dy &\approx \int_0^u (1 - \frac{1}{n} \sum_{i=1}^{n} 1_{(-\infty,y]}(X_i))dy \\ &= \quad u - \frac{1}{n} \sum_{i=1}^{n} (\max(u, X_i) - X_i) \approx 6.98482. \end{aligned}$$

Using the latter, we find that

$$\Psi(45) \approx 0.103752.$$

Problem 4 (i) Recall that

$$\widehat{S}_{i,m+k} = \widehat{f}^{(m)}_{m-i+k-1} \cdot \ldots \cdot \widehat{f}^{(m)}_{m-i} \cdot S_{i,m},$$

where

$$\widehat{f}_j^{(m)} = \frac{\sum_{i=1}^{m-j-1} S_{i,i+j+1}}{\sum_{i=1}^{m-j-1} S_{i,i+j}}$$

is the chain ladder estimator of f_j.

So

$$\widehat{f}_0^{(m)} = \frac{\sum_{i=2016}^{2019} S_{i,i+1}}{\sum_{i=2016}^{2019} S_{i,i}} \approx 1.3649,$$

$$\widehat{f}_1^{(m)} = \frac{\sum_{i=2016}^{2018} S_{i,i+2}}{\sum_{i=2016}^{2018} S_{i,i+1}} \approx 1.24911,$$

$$\widehat{f}_2^{(m)} = \frac{\sum_{i=2016}^{2017} S_{i,i+3}}{\sum_{i=2016}^{2017} S_{i,i+2}} \approx 1.21049,$$

$$\widehat{f}_3^{(m)} = \frac{\sum_{i=2016}^{2016} S_{i,i+4}}{\sum_{i=2016}^{2016} S_{i,i+3}} \approx 1.08685.$$

Hence, we obtain the following predictors in the run-off triangle table:

Year i	$S_{i,i}$	$S_{i,i+1}$	$S_{i,i+2}$	$S_{i,i+3}$	$S_{i,i+4}$
2016	65237	85433	115474	137233	149152
2017	74231	98165	119431	147116	159893
2018	90257	128675	155157	187815	204127
2019	108344	149156	186312	225527	245115
2020	148343	202473	252910	306144	332733

(ii) Using those predictors in (i), we get the technical provision for 2022, i.e.

$$\sum_{j=0}^{2} (\widehat{S}_{2018+j,2022} - \widehat{S}_{2018+j,2021})$$

$$(204127 - 187815) + (225527 - 186312) + (252910 - 202473) = 105964$$

and that of 2024, i.e.

$$(332733 - 306144) = 26589$$

Problem 5 We observe for $i \geq 1$ that

$$\begin{aligned} \{X_i = X_{i+1}\} &= \{\max(Y_{i-2}, Y_{i-1}, Y_i) = \max(Y_{i-1}, Y_i, Y_{i+1})\} \\ &\supseteq \{Y_{i-2} \leq \max(Y_{i-1}, Y_i) \text{ and } \max(Y_{i-1}, Y_i) \geq Y_{i+1}\} \\ &= \{Y_{i-2} \leq Z_i \text{ and } Z_i \geq Y_{i+1}\}, \end{aligned}$$

where $Z_i := \max(Y_{i-1}, Y_i)$. Since $(Y_i)_{i\geq -1}$ is $i.i.d$, we see that Z_i is independent of Y_{i-2}, Y_{i+1}. So it follows from Fubini 's theorem (dobbeltforventning) and the continuity of F that

$$\begin{aligned}
& P(\{Y_{i-2} \leq Z_i \text{ and } Z_i \geq Y_{i+1}\}) \\
= \;& E\left[1_{\{Y_{i-2}\leq Z_i \text{ and } Z_i \geq Y_{i+1}\}}\right] = E\left[E\left[1_{\{Y_{i-2}\leq z \text{ and } z\geq Y_{i+1}\}}(z)\right]\big|_{z=Z_i}\right] \\
= \;& E\left[P(\{Y_{i-2} \leq z \text{ and } z \geq Y_{i+1}\})|_{z=Z_i}\right] = E\left[P(Y_{i-2}\leq z)P(Y_{i+1}\leq z)|_{z=Z_i}\right] \\
= \;& E\left[(P(Y_1 \leq z))^2\big|_{z=Z_i}\right] = E\left[(P(Y_1 \leq z))^2\big|_{z=Z_i}\, 1_{\{Y_i > Y_{i-1}\}}\right] \\
& + E\left[(P(Y_1 \leq z))^2\big|_{z=Z_i}\, 1_{\{Y_i \leq Y_{i-1}\}}\right] \\
= \;& E\left[(P(Y_1 \leq z))^2\big|_{z=Y_i}\, 1_{\{Y_i > Y_{i-1}\}}\right] + E\left[(P(Y_1 \leq z))^2\big|_{z=Y_{i-1}}\, 1_{\{Y_i \leq Y_{i-1}\}}\right] \\
= \;& E\left[E\left[(P(Y_1 \leq z))^2\big|_{z=x}\, 1_{\{x>Y_{i-1}\}}\right]\big|_{x=Y_i}\right] + E\left[E\left[(P(Y_1 \leq z))^2\big|_{z=x}\, 1_{\{Y_i\leq x\}}\right]\big|_{x=Y_{i-1}}\right] \\
= \;& E\left[(P(Y_1 \leq x))^2 E\left[1_{\{x>Y_{i-1}\}}\right]\big|_{x=Y_i}\right] + E\left[(P(Y_1 \leq x))^2 E\left[1_{\{Y_i\leq x\}}\right]\big|_{x=Y_{i-1}}\right] \\
= \;& E\left[(P(Y_1 \leq x))^3\big|_{x=Y_i}\right] + E\left[(P(Y_1 \leq x))^3\big|_{x=Y_{i-1}}\right] \\
= \;& 2E\left[(P(Y_1 \leq x))^3\big|_{x=Y_1}\right] = 2E\left[(G(x))^3\big|_{x=Y_1}\right] \\
= \;& 2E\left[(G(Y_1))^3\right] = 2\int_0^1 y^3 dy = 2/4 = \frac{1}{2}.
\end{aligned}$$

Here we used the fact that the random variable $X := G(Y_1) \sim U(0,1)$ (uniform distribution). The latter can be seen as follows: Since F has a strictly positive probability density f, F must be a strictly increasing function, otherwise there exist $x_1 < x_2$ such that $F(x_1) = F(x_2)$. So

$$0 = F(x_2) - F(x_1) = \int_{x_1}^{x_2} f(y)dy.$$

Hence $f(y) = 0$ on $[x_1, x_2]$ a.e. However, $f(y) > 0$ for all y. So F is strictly increasing on $\mathbb{R}$. Therefore, $G = \sqrt[3]{F}$ has an (increasing) inverse G^{-1} on $(0,1)$ We get that

$$P(a \leq X \leq b) = P(G^{-1}(a) \leq Y_1 \leq G^{-1}(b)) = G(G^{-1}(b)) - G(G^{-1}(a)) = b - a$$

for all $0 < a \leq b < 1$. So $X \sim U(0,1)$.

Altogether, we see that

$$P(N_n \notin M_S(E)) \geq P(\{X_1 = X_2\}) \geq \frac{1}{2}.$$