Non-Life Insurance Mathematics (STK4540)
Solutions to the exam

Problem 1 (i) Since the claim numbers are modelled by a Poisson process, the inter-arrival times $W_i, i \geq 1$ are *i.i.d.* with common distribution $W_1 \sim Exp(\lambda)$. So by Exercises 1, Problem 1 we know that the MLE $\widehat{\lambda}$ is given by

$$\widehat{\lambda} = \frac{n}{\sum_{i=1}^{n} W_i}.$$

The observed arrival times are $W_1 = 3$ (the day 01/29/1980 is excluded), $W_2 = 2, W_3 = 2, W_4 = 3, W_5 = 6, W_6 = 3, W_7 = 1, W_8 = 1, W_9 = 1, W_{10} = 1$. So $n = 10$ (sample size) and we get that

$$\widehat{\lambda} = \frac{10}{23} \approx 0.43478.$$

(ii) We know that

$$\Psi(u) \sim \rho^{-1} \overline{F}^*_{X_1,I}(u)$$

for $u \longrightarrow \infty$ (Th. 3.4.13).

Next we want to approximate

$$F^*_{X_1,I}(u) = \frac{1}{E^*[X_1]} \int_0^u P^*(X_1 > y) dy$$

by means of the empirical distribution function F_n.

So we see that

$$E^*[X_1] \approx \frac{1}{n} \sum_{i=1}^{n} X_i \overset{n=10}{=} 4.78876.$$

On the other hand,

$$P^*(X_1 > y) \approx 1 - F_n(y).$$

Hence,

$$\begin{aligned} \int_0^u P^*(X_1 &> y) dy \approx \int_0^u (1 - \frac{1}{n} \sum_{i=1}^{n} 1_{(-\infty,y]}(X_i)) dy \\ &= u - \frac{1}{n} \sum_{i=1}^{n} (\max(u, X_i) - X_i) \approx 4.3708976. \end{aligned}$$

Using the latter, we find that

$$\Psi(10) \approx 0.436295.$$

Problem 2 By Theorem 6.3.1 in Mikosch we have that

$$\widehat{\mu}_{LB} = (1 - w)\mu + w\overline{Y},$$

where

$$w = \frac{n\lambda}{\varphi + n\lambda} \text{ and } \overline{Y} = \frac{1}{n}\sum_{j=1}^{n} Y_j.$$

By Exercises 5, Prob. 2 we know that

1.

$$\begin{aligned} \mu &= E[p(\theta)] = E[P(X_1 > K\,|\theta)] = E[(\frac{\lambda'}{K})^{\theta}] \\ &= \left(\frac{\beta}{\beta - \log(\lambda'/K)}\right)^{\gamma} \approx 0.132343. \end{aligned}$$

2.

$$\begin{aligned} \lambda &= Var[p(\theta)] = Var[(\frac{\lambda'}{K})^{\theta}] \\ &= \left(\frac{\beta}{\beta - 2\log(\lambda'/K)}\right)^{\gamma} - \left(\frac{\beta}{\beta - \log(\lambda'/K)}\right)^{2\gamma} \approx 0.0306465. \end{aligned}$$

3.

$$\begin{aligned} \varphi &= E[Var[X_1\,|\theta]] = E[p(\theta)] - E[(p(\theta))^2] \\ &= \left(\frac{\beta}{\beta - \log(\lambda'/K)}\right)^{\gamma} - \left(\frac{\beta}{\beta - 2\log(\lambda'/K)}\right)^{\gamma} \approx 0.0841817. \end{aligned}$$

So

$$w \overset{n=10}{=} \frac{10 \cdot 0.0306465}{0.0841817 + 10 \cdot 0.0306465} \approx 0.784507 \text{ and } \overline{Y} = 0.3.$$

Therefore, we get that

$$\widehat{\mu}_{LB} = (1-w)\mu + w\overline{Y} \approx 26.387\%$$

is the estimated probability for $X_1 > 60000$ NOK given θ.

The corresponding risk is given by

$$\rho(\widehat{\mu}_{LB}) = (1-w)\lambda \approx 0.00660411.$$

Problem 3 (i) We apply Theorem 6.3.1 in Mikosch and get that

$$\widehat{\mu}_{LB} = (1-w)\mu + w\overline{X},$$

where

$$w = \frac{n\lambda}{\varphi + n\lambda} \text{ and } \overline{X} = \frac{1}{n}\sum_{j=1}^{n} X_j.$$

Here:

1.

$$\mu = E[\mu(\theta)] = E[\theta] \overset{\text{hint}}{=} \frac{1.5}{1.7} \approx 0.882353.$$

2.

$$\lambda = Var[\mu(\theta)] = Var[\theta] \overset{\text{hint}}{=} \frac{1.5}{(1.7)^2} \approx 0.519031.$$

3.

$$\varphi = E[Var[X_1 \,|\theta]] = E[\theta] \overset{\text{hint}}{=} \frac{1.5}{1.7} \approx 0.882353.$$

Thus

$$w \overset{n=10}{=} \frac{10 \cdot 0.519031}{0.882353 + 10 \cdot 0.519031} \approx 0.854701 \text{ and } \overline{X} = 1.5.$$

Therefore, we get that

$$\widehat{\mu}_{LB} = (1-w)\mu + w\overline{X} \approx 1.41026.$$

expected claim number (per year) given the observations.

The corresponding risk is given by

$$\rho(\widehat{\mu}_{LB}) = (1-w)\lambda \approx 0.0754147.$$

(ii) We now assume in (i) that $\theta = Z^3$ for $Z \sim Exp(1)$. Then

1.

$$\begin{aligned} \mu &= E[\mu(\theta)] = E[Z^3] = \int_0^\infty y^3 \exp(-y)dy \\ &= 3! \int_0^\infty \frac{1}{3!} y^{4-1} \exp(-y)dy = 6. \end{aligned}$$

$$\begin{aligned} \lambda &= Var[\mu(\theta)] = E[(\mu(\theta))^2] - \mu^2 \\ &= E[Z^6] - \mu^2 \\ &= \int_0^\infty y^{7-1} \exp(-y)dy - \mu^2 \\ &= 6! \int_0^\infty \frac{1}{6!} y^{7-1} \exp(-y)dy - \mu^2 \\ &= 6! - \mu^2 = 684. \end{aligned}$$

3.

$$\varphi = E[Var[X_1 \,|\theta]] = E[\theta] = 6.$$

So

$$w \overset{n=10}{=} \frac{10 \cdot 684}{6 + 10 \cdot 684} \approx 0.999124 \text{ and } \overline{X} = 1.5.$$

Hence

$$\widehat{\mu}_{LB} = (1-w)\mu + w\overline{X} = 1.50394.$$

Problem 4 (i) Recall that

$$\widehat{S}_{i,m+k} = \widehat{f}^{(m)}_{m-i+k-1} \cdot \ldots \cdot \widehat{f}^{(m)}_{m-i} \cdot S_{i,m},$$

where

$$\widehat{f}_j^{(m)} = \frac{\sum_{i=1}^{m-j-1} S_{i,i+j+1}}{\sum_{i=1}^{m-j-1} S_{i,i+j}}$$

is the chain ladder estimator of f_j.

So

$$\widehat{f}_0^{(m)} = \frac{\sum_{i=2016}^{2019} S_{i,i+1}}{\sum_{i=2016}^{2019} S_{i,i}} \approx 1.30167,$$

$$\widehat{f}_1^{(m)} = \frac{\sum_{i=2016}^{2018} S_{i,i+2}}{\sum_{i=2016}^{2018} S_{i,i+1}} \approx 1.25966,$$

$$\widehat{f}_2^{(m)} = \frac{\sum_{i=2016}^{2017} S_{i,i+3}}{\sum_{i=2016}^{2017} S_{i,i+2}} \approx 1.21029,$$

$$\widehat{f}_3^{(m)} = \frac{\sum_{i=2016}^{2016} S_{i,i+4}}{\sum_{i=2016}^{2016} S_{i,i+3}} \approx 1.13549.$$

Hence, we obtain the following predictors in the run-off triangle table:

Year i	$S_{i,i}$	$S_{i,i+1}$	$S_{i,i+2}$	$S_{i,i+3}$	$S_{i,i+4}$
2017	66277	81433	112473	132234	150151
2018	78232	96163	117431	146016	165800
2019	94259	129679	157158	190206	215978
2020	110344	147155	185365	224345	254743
2021	149348	194402	244881	296376	336534

(ii) Using those predictors in (i), we get the technical provision for 2023, i.e.

$$\sum_{j=0}^{2} (\widehat{S}_{2019+j,2023} - \widehat{S}_{2019+j,2022})$$

$$(215978 - 190206) + (224345 - 185365) + (244881 - 194402) = 115231$$

and that of 2025, i.e.

$$(336534 - 296376) = 40158.$$

Problem 5 Consider first the case $n = 2$ and assume $X_1 \sim Exp(\lambda)$ for a constant $\lambda > 0$. Define $L(x_1, x_2) = (\frac{x_2}{2}, \frac{x_2}{2} + \frac{x_1}{1}) = (\frac{x_2}{2}, \frac{x_2}{2} + x_1)$. Then one observes that the function L has an inverse L^{-1} given by $L^{-1}(y_1, y_2) = (y_2 - y_1, 2y_1)$. Taking into account that

$$\begin{aligned}
& P(X_{(1)} \leq x_1, X_{(2)} \leq x_2) \\
= \; & P(X_{(1)} \leq x_1, X_{(2)} \leq x_2, X_1 < X_2) + P(X_{(1)} \leq x_1, X_{(2)} \leq x_2, X_1 > X_2) \\
= \; & P(X_1 \leq x_1, X_2 \leq x_2, X_1 < X_2) + P(X_2 \leq x_1, X_1 \leq x_2, X_1 > X_2) \\
= \; & 2P(X_1 \leq x_1, X_2 \leq x_2, X_1 < X_2)
\end{aligned}$$

we get

$$\begin{aligned}
P(X_{(1)} &\leq x_1, X_{(2)} \leq x_2) = \int_0^{x_1}\int_0^{x_2} f_{X_{(1)},X_{(2)}}(y_1,y_2)dy_1dy_2 \\
&= \int_0^{x_1}\int_0^{x_2} 2! f_{X_1}(y_1)f_{X_2}(y_2)1_{\{y_2>y_1\}}dy_1dy_2 \\
&\quad \int_0^{x_1}\int_0^{x_2} 2\lambda e^{-\lambda(y_2-y_1)}\lambda e^{-\lambda(2y_1)}1_{\{y_2>y_1\}}dy_1dy_2 \\
&\quad \overset{X_1,X_2 \text{ indep.}}{=} \int_0^{x_1}\int_0^{x_2} 2\cdot f_{X_1,X_2}(L^{-1}(y_1,y_2))1_{\{y_2>y_1\}}dy_1dy_2 \\
&\quad \overset{\text{substitution.:}(u_1,u_2)=L^{-1}(y_1,y_2)}{=} \int_{L^{-1}((0,x_1)\times(0,x_2))} f_{X_1,X_2}(u_1,u_2)du_1du_2 \\
&= P((X_1,X_2)\in L^{-1}([0,x_1]\times[0,x_2])) \\
&= P(L(X_1,X_2)\in[0,x_1]\times[0,x_2]) = P(\frac{X_2}{2}\leq x_1, \frac{X_2}{2}+\frac{X_1}{1}\leq x_2).
\end{aligned}$$

So

$$(X_{(1)},X_{(2)}) \overset{d}{=} (\frac{X_2}{2}, \frac{X_2}{2}+\frac{X_1}{1}).$$

In the same way one also shows more generally that

$$(X_{(1)},...,X_{(n)}) \overset{d}{=} (\frac{X_n}{n}, \frac{X_n}{n}+\frac{X_{n-1}}{n-1},..., \frac{X_n}{n}+\frac{X_{n-1}}{n-1}+...+\frac{X_2}{2}, \frac{X_n}{n}+\frac{X_{n-1}}{n-1}+...+\frac{X_1}{1}). \quad (1)$$

Define

$$G(x_1,...,x_n) = \sum_{i=1}^{k-1} x_{n-i+1} - (k-1)\cdot x_{n-k+1}.$$

Then it follows from (1) for all $n \geq k \geq 2$ that

$$\begin{aligned}
\sum_{i=1}^{n}(X_{(n-i+1)} - X_{(n-k+1)+} &= \sum_{i=1}^{k-1} X_{(n-i+1)} - (k-1)\cdot X_{(n-k+1)} = G(X_{(1)},...,X_{(n)}) \\
\overset{d}{=} G(\frac{X_n}{n},..., \frac{X_n}{n}+\frac{X_{n-1}}{n-1}+...+\frac{X_1}{1}) &= X_1+...+X_{k-1} \sim \Gamma(k-1,\lambda) \text{ (see Exerc. 1, Prob. 1).}
\end{aligned}$$

Suppose now that $X_1 \sim Exp(\theta)$. Then

$$\begin{aligned} P(S(t) &\leq x \,|N(t) \geq k) = \\ &\sum_{n\geq k} E\left[1_{\{S(t)\leq x\}} 1_{\{N(t)=n\}} \,|N(t) \geq k\right] \\ &= \sum_{n\geq k} E\left[1_{\left\{G(X_{(1)},\ldots,X_{(n)})\leq x\right\}} 1_{\{N(t)=n\}} \,|N(t) \geq k\right] \\ &\overset{\text{independence}}{=} \\ P(G(X_{(1)},...,X_{(n)}) &\leq x) \sum_{n\geq k} P(N(t)=n \,|N(t)\geq k) \\ &= E\left[P(G(X_{(1)},...,X_{(n)}) \leq x \,|\theta)\right] \sum_{n\geq k} P(N(t)=n \,|N(t)\geq k) \\ &= E\left[P(X_1+...+X_{k-1} \leq x \,|\theta)\right] \sum_{n\geq k} P(N(t)=n \,|N(t)\geq k) \\ &= E\left[P(X_1+...+X_{k-1} \leq x \,|\theta)\right]. \end{aligned}$$

where

$$P(X_1+...+X_{k-1} \leq x \,|\theta) = 1-\exp(-\theta x)\sum_{j=0}^{k-2}\frac{(\theta x)^j}{j!}.$$

Hence

$$\begin{aligned} P(S(t) &\leq x \,|N(t) \geq k) = \\ &= E\left[1-\exp(-\theta x)\sum_{j=0}^{k-2}\frac{(\theta x)^j}{j!}\right] \\ &= 1-\int_0^\infty \exp(-xy)\sum_{j=0}^{k-2}\frac{(yx)^j}{j!}\frac{\beta^\gamma}{\Gamma(\gamma)}y^{\gamma-1}e^{-\beta y}dy \\ &= 1-\sum_{j=0}^{k-2}\frac{x^j}{j!}\frac{\beta^\gamma}{\Gamma(\gamma)}\frac{\Gamma(\gamma+j)}{(\beta+x)^{\gamma+j}}\int_0^\infty \exp(-(x+\beta)y)y^{\gamma+j-1}\frac{(\beta+x)^{\gamma+j}}{\Gamma(\gamma+j)}dy \\ &= 1-\sum_{j=0}^{k-2}\frac{x^j}{j!}\frac{\beta^\gamma}{\Gamma(\gamma)}\frac{\Gamma(\gamma+j)}{(\beta+x)^{\gamma+j}} \end{aligned}$$

(ii) So for $\gamma=2, \beta=1, x=10, k=5, t=2$ we get

$$\begin{aligned} P(S(t) &\leq 10 \,|N(t)\geq 5) = 1-\sum_{j=0}^{3}\frac{10^j}{j!}\frac{1}{1}\frac{(2+j-1)!}{(11)^{\gamma+j}} \\ &= 1-\frac{1}{11^2}-10\frac{2}{11^3}-\frac{10^2}{2}\frac{6}{11^4}-\frac{10^3}{6}\frac{24}{11^5} \\ &\approx 0.9314. \end{aligned}$$