UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Examination in	STK4540 — Non-Life Insurance Mathematics.
Day of examination:	November 28, 2023, Final written exam.
Examination hours:	15:00 – 19:00.
This problem set consists of 3 pages.	
Appendices:	None
Permitted aids:	Approved calculator

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1. (10 points)

Consider the heterogeneity model for the i-th policy of an insurance. Given $\theta \sim \Gamma(\gamma, \beta)$ (Gamma distribution) the claim sizes $X_1, ..., X_n$ in the policy are Pareto distributed with parameters (λ, θ), that is

$$P(X_i > x \,|\theta) = (\lambda/x)^\theta, x > \lambda.$$

A reinsurance company only considers the values X_i which exceed a known high treshold K. They "observe" the counting variables $Y_i := 1_{(K,\infty)}(X_i)$ for a known $K > \lambda$. The company wants to find an estimate of $p(\theta) := P(X_1 > K \,|\theta)$.

Calculate the linear Bayes estimator and its risk with respect to $p(\theta) := E[Y_1 \,|\theta]$ based on the observations $Y_1, ..., Y_n$, when $K = 50$, $\lambda = 1.8$, $X_1 = 25$, $X_5 = 85$, $X_{10} = 75$ and $X_j = 2$ (in 1000 NOK) for $j = 2, 3, 4, 6, 7, 8, 9$ (in years). Further, suppose that $\beta = \gamma = 2.5$.

Hint: Recall that the Gamma distribution has the probability density

$$f_{\gamma,\beta}(x) = \frac{\beta^\gamma}{\Gamma(\gamma)} x^{\gamma-1} e^{-\beta x}, x > 0,$$

where $\Gamma(z)$ is the Gamma function. Here $\Gamma(n) = (n-1)!$ for $n \geq 1$.

Problem 2. (10 points)

Assume again the heterogeneity model and suppose that in the i-th policy of an insurance the claim numbers $X_t, t \geq 1$ given the heterogeneity parameter θ are Poisson distributed with intensity θ.

(i) Assume that θ is Gamma distributed with parameters $\gamma, \beta > 0$.

(Continued on page 2.)

Calculate the linear Bayes estimator $\widehat{\mu}_{LB}$ of the expected claim number $\mu(\theta) := E[X\,|\theta]$ and its corresponding risk for the claim number data $X_1 = 3$, $X_4 = 1, X_5 = 1$, $X_7 = 8, X_{10} = 1$ and $X_j = 0$ (in 1000 NOK) for $j = 2, 3, 6, 8, 9$ (in years) and the parameters $\gamma = 1.9, \beta = 1.2$.
(ii) Assume now that $\theta = Z^2$, where $Z \sim Exp(\lambda^*)$ (exponential distribution) for $\lambda^* = 1$.
Compute $\widehat{\mu}_{LB}$ for the same data as in (i).

Hint: $E[\theta] = \frac{\gamma}{\beta}$, $Var[\theta] = \frac{\gamma}{\beta^2}$, if $\theta \sim \Gamma(\gamma, \beta)$.

Problem 3. (10 points)

Let us have a look at the following Danish fire insurance data collected at Copenhagen Reinsurance in the period from 07/30/1988 until 08/24/1988:

Date	Loss X_i in DKM	Date	Loss X_i in DKM
08/08/1988	1.330967	08/18/1988	2.309672
08/09/1988	1.774623	08/21/1988	1.145519
08/10/1988	3.012422	08/21/1988	1.376220
08/12/1988	47.019521	08/22/1988	1.142857
08/15/1988	1.100266	08/24/1988	4.670808

The fire losses in the above table are given in millions of Danish Krone. Suppose that the dynamics of the total claim amount $S(t), t \geq 0$ is modelled by the Crámer-Lundberg model.

(i) Compute the maximum-likelihood estimator $\widehat{\lambda}$ of the intensity of the claim number process $N(t), t \geq 0$.
(ii) Let $F^* = F^*_{X_1}$ be the real claim size distribution. Suppose that the net profit condition (NPC) holds and that $E^*[X_1] < \infty$ (expected value of X_1 with respect to F^*). Further, require that F^* has a probability density and that its integrated tail distribution $F^*_{X_1,I}$ is subexponential.
Calculate a (rough) estimate of the ruin probability $\Psi(u)$ for the safety loading $\rho = 30\%$ and the initial capital $u = 45$ (in millions of Danish Krone) by approximating F^* by means of the empirical distribution function with respect to the fire insurance data above.

Hint: Apply the following convention: Claims $X_1^{(m)}, ..., X_n^{(m)}$ arriving on day $m \geq 0$ are interpreted as 1 claim $X_m := \sum_{i=1}^n X_i^{(m)}$ arriving on day m.

Problem 4. (10 points)

Consider Mack's model. The following *run-off triangle* in the figure shows the cumulative paid claims:

Year i	$S_{i,i}$	$S_{i,i+1}$	$S_{i,i+2}$	$S_{i,i+3}$	$S_{i,i+4}$
2016	65237	85433	115474	137233	149152
2017	74231	98165	119431	147116	
2018	90257	128675	155157		
2019	108344	149156			
2020	148343				

(Continued on page 3.)

(i) Compute the predictors $\widehat{S}_{i,m+k}$ of the total claim amounts

$$\begin{aligned}
&S_{2017,m+1},\\
&S_{2018,m+1}, S_{2018,m+2},\\
&S_{2019,m+1}, S_{2019,m+2}, S_{2019,m+3},\\
&S_{2020,m+1}, S_{2020,m+2}, S_{2020,m+3}, S_{2020,m+4}.
\end{aligned}$$

for $m = 2020$ based on the data of the run-off triangle.
(ii) Calculate the technical provisions for the year 2022, that is

$$\sum_{j=0}^{2}(\widehat{S}_{2018+j,2022} - \widehat{S}_{2018+j,2021})$$

and for the year 2024, that is

$$\widehat{S}_{2020,2024} - \widehat{S}_{2020,2023}.$$

Hint: Recall that

$$\widehat{S}_{i,m+k} = \widehat{f}^{(m)}_{m-i+k-1} \cdot \ldots \cdot \widehat{f}^{(m)}_{m-i} \cdot S_{i,m},$$

where

$$\widehat{f}^{(m)}_{j} = \frac{\sum_{i=1}^{m-j-1} S_{i,i+j+1}}{\sum_{i=1}^{m-j-1} S_{i,i+j}}$$

is the chain ladder estimator of f_j.

Problem 5. (10 points)

Let F be a distribution function on $\mathbb{R}$ with a strictly positive probability density f. Consider an $i.i.d$-sequence of random variables $Y_i, i = -1, 0, 1, 2, 3, ...$ Assume that the distribution function G of Y_i is given by $G(x) = \sqrt[3]{F(x)}$ for all x.
Denote by $M_p(E)$ the space of point measures on $E = \mathbb{R}$ and denote by $M_S(E)$ the space of simple point measures on $E = \mathbb{R}$, that is

$$M_S(E) = \{\mu \in M_p(E) : \mu(\{x\}) \leq 1 \text{ for all } x \in E\}.$$

Consider the point process $N_n : \Omega \longrightarrow M_p(E)$ defined by

$$N_n = \sum_{i=1}^{n} \varepsilon_{X_i}$$

for $n \geq 2$, where $X_i := \max(Y_{i-2}, Y_{i-1}, Y_i), i \geq 1$ and ε_x is the Dirac measure in x.
Show that

$$P(N_n \notin M_S(E)) \geq \frac{1}{2}.$$

End