UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Examination in	STK4540 — Non-Life Insurance Mathematics.
Day of examination:	December 17, 2024, Final written exam.
Examination hours:	09:00 – 13:00.
This problem set consists of 4 pages.	
Appendices:	None
Permitted aids:	Approved calculator

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1. (10 points)

Consider the following Danish fire insurance data collected at Copenhagen Reinsurance in the period from 01/29/1980 until 02/23/1980:

Date	Loss X_i in DKM	Date	Loss X_i in DKM
02/03/1980	2.002430	02/19/1980	11.713031
02/05/1980	4.530015	02/20/1980	1.515373
02/07/1980	1.841753	02/21/1980	2.538589
02/10/1980	3.806735	02/22/1980	2.049780
02/16/1980	5.424253	02/23/1980	12.465593

The fire losses in the above table are given in millions of Danish Krone. Suppose that the dynamics of the total claim amount $S(t), t \geq 0$ is modelled by the Crámer-Lundberg model.

(i) Compute the maximum-likelihood estimator $\widehat{\lambda}$ of the intensity of the claim number process $N(t), t \geq 0$.

(ii) Let $F^* = F^*_{X_1}$ be the real claim size distribution. Suppose that the net profit condition (NPC) holds and that $E^*[X_1] < \infty$ (expected value of X_1 with respect to F^*). Further, require that F^* has a probability density and that its integrated tail distribution $F^*_{X_1,I}$ is subexponential.

Calculate a (rough) estimate of the ruin probability $\Psi(u)$ for the safety loading $\rho = 20\%$ and the initial capital $u = 10$ (in millions of Danish Krone) by approximating F^* by means of the empirical distribution function with respect to the fire insurance data above.

Problem 2. (10 points)

Assume the heterogeneity model for the i-th policy of an insurance. Given $\theta \sim \Gamma(\gamma, \beta)$

(Continued on page 2.)

(Gamma distribution) the claim sizes $X_1, ..., X_n$ in the policy are Pareto distributed with parameters (λ, θ), that is

$$P(X_i > x \,|\theta) = (\lambda/x)^\theta, x > \lambda.$$

A reinsurance company only takes into account the values X_i which exceed a known high treshold K. They "observe" the counting variables $Y_i := 1_{(K,\infty)}(X_i)$ for a known $K > \lambda$. The company wants to estimate $p(\theta) := P(X_1 > K \,|\theta)$.

Compute the linear Bayes estimator and its risk with respect to $p(\theta) := E[Y_1 \,|\theta]$ based on the observations $Y_1, ..., Y_n$, when $K = 60$, $\lambda = 2$, $X_1 = 70$, $X_5 = 85$, $X_{10} = 75$ and $X_j = 3$ (in 1000 NOK) for $j = 2, 3, 4, 6, 7, 8, 9$ (in years) . In addition, suppose that $\beta = \gamma = 2.1$.

Hint: Recall that the Gamma distribution has the probability density

$$f_{\gamma,\beta}(x) = \frac{\beta^\gamma}{\Gamma(\gamma)} x^{\gamma-1} e^{-\beta x}, x > 0,$$

where $\Gamma(z)$ is the Gamma function. Here $\Gamma(n) = (n-1)!$ for $n \geq 1$.

Problem 3. (10 points)

Let us again consider the heterogeneity model and assume that in the i-th policy of an insurance the claim numbers $X_t, t \geq 1$ given the heterogeneity parameter θ are Poisson distributed with intensity θ.

(i) Suppose that θ is Gamma distributed with parameters $\gamma, \beta > 0$.

Compute the linear Bayes estimator $\widehat{\mu}_{LB}$ of the expected claim number $\mu(\theta) := E[X \,|\theta]$ and its corresponding risk for the claim number data $X_1 = 5$, $X_4 = 1, X_5 = 2$, $X_7 = 6, X_{10} = 1$ and $X_j = 0$ for $j = 2, 3, 6, 8, 9$ (in years) and the parameters $\gamma = 1.5, \beta = 1.7$.

(ii) Suppose now that $\theta = Z^3$, where $Z \sim Exp(\lambda^*)$ (exponential distribution) for $\lambda^* = 1$. Calculate $\widehat{\mu}_{LB}$ for the same data as in (i).

Hint: $E[\theta] = \frac{\gamma}{\beta}$, $Var[\theta] = \frac{\gamma}{\beta^2}$, if $\theta \sim \Gamma(\gamma, \beta)$.

Problem 4. (10 points)

Assume Mack's model. The following *run-off triangle* in the figure shows the cumulative paid claims:

Year i	$S_{i,i}$	$S_{i,i+1}$	$S_{i,i+2}$	$S_{i,i+3}$	$S_{i,i+4}$
2017	66277	81433	112473	132234	150151
2018	78232	96163	117431	146016	
2019	94259	129679	157158		
2020	110344	147155			
2021	149348				

(Continued on page 3.)

(i) Calculate the predictors $\widehat{S}_{i,m+k}$ of the total claim amounts

$$\begin{aligned}
&S_{2018,m+1},\\
&S_{2019,m+1}, S_{2019,m+2},\\
&S_{2020,m+1}, S_{2020,m+2}, S_{2020,m+3},\\
&S_{2021,m+1}, S_{2021,m+2}, S_{2021,m+3}, S_{2021,m+4}.
\end{aligned}$$

for $m = 2021$ based on the data of the run-off triangle.

(ii) Compute the technical provisions for the year 2023, that is

$$\sum_{j=0}^{2}(\widehat{S}_{2019+j,2023} - \widehat{S}_{2019+j,2022})$$

and for the year 2025, that is

$$\widehat{S}_{2021,2025} - \widehat{S}_{2021,2024}.$$

Hint: Recall that

$$\widehat{S}_{i,m+k} = \widehat{f}^{(m)}_{m-i+k-1} \cdot ... \cdot \widehat{f}^{(m)}_{m-i} \cdot S_{i,m},$$

where

$$\widehat{f}^{(m)}_{j} = \frac{\sum_{i=1}^{m-j-1} S_{i,i+j+1}}{\sum_{i=1}^{m-j-1} S_{i,i+j}}$$

is the chain ladder estimator of f_j.

Problem 5. (10 points)

Assume a homogeneous Poisson process $N(t), t \geq 0$ with jump intensity $\lambda^* > 0$. Further, given $\theta \sim \Gamma(\gamma, \beta)$ the claim sizes $X_i, i \geq 1$ are $i.i.d$ with $X_1 \sim Exp(\theta)$, that is

$$P(X_1 \leq x \,|\theta) = 1 - \exp(-\theta x),\ x \geq 0.$$

Suppose that $X_i, i \geq 1$ and $N(t), t \geq 0$ are independent. Denote by $X_{(j)}$ the $j-$th smallest claim size of the claim sizes $X_1, ..., X_n$. So $X_{(1)} \leq ... \leq X_{(n)}$. Consider now the claim size amount $S(t)$ at time t given by the losses in excess of the k-th largest claim size, that is consider

$$S(t) = \sum_{i=1}^{N(t)}(X_{(N(t)-i+1)} - X_{(N(t)-k+1)})_+$$

for $N(t) \geq k$ and $k \geq 2$, where $(a)_+ \overset{def}{=} \max(a, 0)$ for $a \in \mathbb{R}$.

(i) Derive a formula for the distribution

$$P(S(t) \leq x \,|N(t) \geq k)$$

(Continued on page 4.)

for $k \geq 2$.

(ii) Compute $P(S(t) \leq 10 \,|N(t) \geq 5)$ for $\lambda^* = 1$, $t = 2$, $\gamma = 2$, $\beta = 1$.

End