

Non-Life Insurance Mathematics (STK4540)

Prøveeksamen

Problem 1 Consider the following Danish fire insurance data collected at Copenhagen Reinsurance in the period from 09/30/1990 until 10/10/1990:

<u>Date</u>	<u>Loss X_i in DKM</u>	<u>Date</u>	<u>Loss X_i in DKM</u>
10/01/1990	1.023102	10/04/1990	2.145215
10/01/1990	4.702970	10/04/1990	1.155116
10/02/1990	3.135314	10/07/1990	1.056106
10/03/1990	1.864686	10/08/1990	144.657591
10/03/1990	2.037954	10/10/1990	28.630363

The fire losses in the above table are given in millions of Danish Krone. Assume that the dynamics of the total claim amount $S(t) = \sum_{i=1}^{N(t)} X_i$ is described by the Crámer-Lundberg model.

(i) Compute the maximum-likelihood estimator $\hat{\lambda}$ of the jump intensity of the claim number process $N(t)$.

(ii) Denote by $F^* = F_{X_1}^*$ the real claim size distribution. Assume that the net profit condition (NPC) holds and that $E^*[X_1] < \infty$ (expected value of X_1 with respect to F^*). Further, require that F^* has a probability density and that its integrated tail distribution $F_{X_1, I}^*$ is subexponential.

Give a (rough) estimate of the ruin probability $\Psi(u)$ for the safety loading $\rho = 30\%$ and the initial capital $u = 100$ (in millions of Danish Krone) by approximating F^* by the empirical distribution function with respect to the fire insurance data above.

Hint: Apply the following convention: Claims $X_1^{(m)}, \dots, X_n^{(m)}$ arriving on day $m \geq 0$ are interpreted as 1 claim $X_m := \sum_{i=1}^n X_i^{(m)}$ arriving on day m .

Problem 2 Assume the heterogeneity model and require that in the i -th policy of a car insurance the claim numbers $X_t, t \geq 1$ given the heterogeneity parameter θ are Poisson distributed with intensity θ .

(i) Suppose that θ is Gamma distributed with parameters $\gamma, \beta > 0$.

Calculate the linear Bayes estimator $\hat{\mu}_{LB}$ of the expected claim number $\mu(\theta) := E[X | \theta]$ and its corresponding risk for the claim number data $X_1 = 4, X_5 = 7, X_7 = 3, X_{10} = 1$ and $X_j = 0$ for $j = 2, 3, 4, 6, 8, 9$ (in years) and the parameters $\gamma = 2.9, \beta = 2$.

(ii) Assume now that $\theta \sim U(0, 1)$ (uniform distribution on $(0, 1)$).

Compute $\hat{\mu}_{LB}$ for the same data as in (i).

Hint: Recall that $E[\theta] = \frac{\gamma}{\beta}, \text{Var}[\theta] = \frac{\gamma}{\beta^2}$, if $\theta \sim \Gamma(\gamma, \beta)$.

Problem 3 Let us have a look at the heterogeneity model for the i -th policy of an insurance. Given $\theta \sim \Gamma(\gamma, \beta)$ (Gamma distribution) the claim sizes $X_1, \dots, X_n$ in the policy are assumed to be Pareto distributed with parameters (λ, θ) , that is

$$P(X_i > x | \theta) = (\lambda/x)^\theta, x > \lambda.$$

A reinsurance company takes into account only the values X_i exceeding a known high threshold K . They "observe" the counting variables $Y_i := 1_{(K,\infty)}(X_i)$ for a known $K > \lambda$. The company is interested in estimating $p(\theta) := P(X_1 > K | \theta)$.

Calculate the linear Bayes estimator and its risk with respect to $p(\theta) := E[Y_1 | \theta]$ based on the observed data $Y_1, \dots, Y_n$, when $\lambda = 1, K = 10, X_1 = 10, X_5 = 20, X_{10} = 50, X_j = 1$ (in 1000 NOK) for $j = 2, 3, 4, 6, 7, 8, 9$ (in years). Suppose that $\beta = \gamma = 2$.

Problem 4 Consider the i -th policy in a heterogeneity model. Assume that the claim size variables $X_j, j \geq 1$ given θ are log-normally distributed, that is $X_j = \exp(\theta + \tau Z_j), j \geq 1$. Here θ is normally distributed with mean μ and variance σ^2 , $\tau > 0$ is a constant and $Z_j, j \geq 1$ is an *i.i.d.*-sequence of standard normally distributed random variables being independent of θ .

Compute the linear Bayes estimator with respect to the premium of an insurance and for the claim size data $X_1 = 20, X_5 = 100, X_j = 1$ (in 1000 NOK) for $j = 2, 3, 4, 6, 7, 8$ (in years) and the parameters $\mu = 3, \sigma^2 = \tau = \frac{1}{2}$.

Problem 5 Assume the heterogeneity model for an insurance company with several policies. We will focus on one given policy. Suppose a claim history of this policy is described by the sequence $X_1, \dots, X_n, n \geq 1$ of random variables, which are exponentially distributed given θ . Here θ is the heterogeneity parameter following a Gamma distribution with shape parameter α and scale parameter $\beta > 0$.

(i) Compute the Bayes estimator of the net premium $\mu(\theta) = E[X_1 | \theta]$ and its corresponding risk for $\alpha = 3, \beta = 1$ and data $X_3 = 10, X_5 = 20, X_7 = 5, X_{10} = 30, X_j = 1, j = 1, 2, 4, 6, 8, 9$ (in 1000 NOK per year).

(ii) Calculate the corresponding linear Bayes estimator with respect to (i).

Problem 6 Consider Mack's model. The following *run-off triangle* in the figure shows the cumulative paid claims:

Year i	$S_{i,i}$	$S_{i,i+1}$	$S_{i,i+2}$	$S_{i,i+3}$	$S_{i,i+4}$	$S_{i,i+5}$
2011	52546	81275	90461	98277	103162	106264
2012	62285	98495	110096	118346	123682	
2013	72173	113299	124340	132883		
2014	86135	127359	138409			
2015	97068	150476				
2016	128982					

(i) Compute the predictors $\hat{S}_{i,m+k}$ of the total claim amounts

$$\begin{aligned}
&S_{2012,m+1}, \\
&S_{2013,m+1}, S_{2013,m+2}, \\
&S_{2014,m+1}, \dots, S_{2014,m+3}, \\
&S_{2015,m+1}, \dots, S_{2015,m+4}, \\
&S_{2016,m+1}, \dots, S_{2016,m+5}
\end{aligned}$$

for $m = 2016$ based on the data of the run-off triangle.

(ii) Calculate the technical provisions for the year 2018, that is

$$\sum_{j=0}^3 (\hat{S}_{2013+j,2018} - \hat{S}_{2013+j,2017})$$

and for the year 2021, that is

$$\hat{S}_{2016,2021} - \hat{S}_{2016,2020}.$$

Hint: Recall that

$$\hat{S}_{i,m+k} = \hat{f}_{m-i+k-1}^{(m)} \cdot \dots \cdot \hat{f}_{m-i}^{(m)} \cdot S_{i,m},$$

where

$$\hat{f}_j^{(m)} = \frac{\sum_{i=1}^{m-j-1} S_{i,i+j+1}}{\sum_{i=1}^{m-j-1} S_{i,i+j}}$$

is the chain ladder estimator of f_j .