

STK4540: Non-life Insurance Mathematics

Exercise list 5

Exercise 1

Let $X_1, \dots, X_n \sim N(\mu, \sigma^2)$ be a sample of independent identically distributed *a priori* observations where σ^2 is an unknown parameter and $\mu \sim N(\mu_0, \tau^2)$ being μ_0 and τ^2 unknown parameters.

- Find a proportional (with respect to μ) function to the likelihood function. That is $f(\mu) \propto L(x_1, \dots, x_n | \mu)$ being L the likelihood function.
- Write $f(\mu) \cdot \pi(\mu)$ being π the prior and work out a proportional function with respect to μ . That is find $g(\mu) \propto f(\mu)\pi(\mu)$. Hint: Try to merge exponentials and complete squares in the exponents.
- You have actually found a proportional density to the posterior mean μ . Find $p(\mu | x_1, \dots, x_n)$. What distribution do you recognize?
- Find the posterior mean and variance. You should observe that the posterior mean is a weighted sum between the (frequentist) sample mean and the prior mean. Interpret this.
- Let $\mu_0 = 0$ $\tau^2 = 1$ and we estimate σ^2 based on our sample to be $\hat{\sigma}^2 = 0.5$ (in other words, assume $\sigma^2 = 0.5$). Generate a random sample of $x_1, \dots, x_n$ for a sample size n of your choice for some given different values of μ (e.g. $\mu = -1, 0, 0.5, 1, 2$) and plot the prior and posterior distributions for each case together. Comment.

Exercise 2 (Frequentist vs Bayesian)

When running a series of trials we need a rule on when to stop. Two common rules are:

- Run exactly n trials and stop.
- Run trials until you see a certain result and then stop.

In this example we will consider two coin tossing experiments.

Experiment 1: Toss the coin exactly 6 times and report the number of heads.

Experiment 2: Toss the coin until the first tails and report the number of heads.

Marie is worried that her coin is biased towards heads, so before using it in class she tests it for fairness. She runs an experiment and reports to you that her sequence of tosses was *HHHHHT*. But you are only half-listening, and you forget which experiment Marie ran to produce the data.

Frequentist approach

- (a) Since you were not listening, you decide to compute the P -values for both experiments based on the given data. Let θ be the probability of heads. You want to test the following hypotheses

$$H_0 : \theta = 0.5 \quad vs. \quad H_a : \theta > 0.5$$

Compute the P -values in both experiments and report a conclusion. You may test at 5% significance.

- (b) Comment on the results. What has happened?

Bayesian approach

You decide to run a Bayesian analysis instead. This has the advantage that it does not matter which of the two experiments Marie ran, since the binomial and geometric likelihood functions (columns) for the data $HHHHHT$ are proportional. In either case, you must make up a prior, and you choose $Beta(3, 3)$. Your argument is that this is a relatively flat prior concentrated mostly over the interval $0.25 \leq \theta \leq 0.75$. So it seems a good naive prior.

- (a) Based on the observed data, find the posterior distribution of θ . Plot both the prior and posterior distributions of θ (before and after collecting data) and compare. (You can use R-program)
- (b) Compute the MSE Bayes estimate. Compute also the 50% and 90% credible intervals.
- (c) What is the probability that $\theta > 0.5$ after empirical observation? What is then your final conclusion?