Extreme Value Statistics and Large Deviations

0. Introduction:

Basic literature:

1. Embrechts, P., Klüppelberg, C., Mikosch, T.: Modelling Extremal Events: for Insurance and Finance. Springer (2012).
2. Varadhan, S.R.S.: Large Deviations and Applications. Courant Institute of Mathematical Sciences (1994/2016).

! Sometimes we will also consult

3. McNeil, A., Frey, R., Embrechts, P.: Quantitative Risk Management. Princeton Series in Finance (2015).

What is extreme value statistics (or extreme value theory) about?

→ Roughly speaking, based on an ordered sample of a given random variable, extreme value theory deals with the assessment of the probability of events that are more extreme than previously observed ones.

Here, a central issue is the mathematical analysis and description of rare events or events, which occur with small probability, but have a significant impact on the behaviour of stochastic models of phenomena in real-life applications in e.g. physics, engineering, finance or insurance.

For example, consider the Cramér-Lundberg-model for the description of the dynamics of the total claim amount $S(t)$ at time t of an insurance portfolio:

$$S(t) = \sum_{i=1}^{N(t)} X_i = X_1 + X_2 + \dots + X_{N(t)}, \qquad (0.1)$$

where

(i) $X_i, i \geq 1$ is an i.i.d-sequence of non-negative random variables. Here X_i stands for the i-th claim (e.g. fire loss)

(ii) $N(t), t \geq 0$ standard Poisson process, which models the number of claims until time t

In many applications (e.g. wind storm insurance) there is empirical evidence that the distributions of X_i are subexponential (or heavy-tailed), that is

$$\lim_{x \to \infty} \frac{P(X_1 + \dots + X_n > x)}{P(\max(X_1, \dots, X_n) > x)} = 1 \qquad (0.2)$$

for all $n \geq 2$.

→ tails of the distribution of the sum of claims are asymptotically of the same order as the tails of the distribution of the most extremal claims in the sense of the largest claims

→ indicates that the largest claim has a strong impact on the size of $S(t)$, which is crucial for the calculation of insurance premiums and the solvency of insurance companies

→ Typically, extreme value theory deals with the study of extremal values of samples e.g. of the form

$$M_n := \max(X_1, \dots, X_n)$$

or

$$m_n := \min(X_1, \dots, X_n)$$

→ other examples:

(i) Dam or dike at a river must be built high enough to exceed the maximum water height
→ If X_i models the annual maximal water height in year i, then one is less interested here in $E[X_i]$, but in the distr. of M_n

(ii) Air pollution monitoring stations at various sites of a city: Pollution concentrations (e.g. CO) measured at each site must be below certain levels
→ If X_i models the daily maximal concentration of CO at a certain site of the city, then the knowledge of the distributional behaviour of M_n is important

(iii) Athletic records (e.g. in running)

(iv) Mechanical components such as an airplane wing: Design must be such that the airplane wing withstands stresses from several sources

(v) Equity risks: Day to day market risk

In theory, the exact distr. of M_n can be directly calculated as follows, if $X_i, i \geq 1$ is an i.i.d.-sequence of random variables:

$$P(M_n \leq x) = P(X_1 \leq x, \dots, X_n \leq x) \overset{X_i \text{ indep.}}{=} \prod_{i=1}^{n} P(X_i \leq x) \overset{X_i \text{ identically distr.}}{=} (F(x))^n,$$

where $F(x) := P(X_1 \leq x)$

In practice, however, the distr. function F

is in general not known.

→ possible solution to this problem:
central result of extreme value theory!
An asymptotic result given by the
Fisher-Tippet-Gnedenko theorem,
which can be regarded as a sort of
central limit theorem for extreme values

→ Theorem (limit laws for maxima)

Let X_i, $i \geq 1$ be a sequence of i.i.d. random variables. Assume that there exist norming constants $c_n > 0$, $d_n \in \mathbb{R}$ and a non-degenerate distr. function H (i.e. H has more than two values) such that

$$P(c_n^{-1}(M_n - d_n) \leq x) \xrightarrow[n \to \infty]{} H(x) \qquad (0.3)$$

for all continuity points x of H.

Then H is given by one of the following distr. functions:

1. Fréchet: $\Phi_\alpha(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ \exp(-x^{-\alpha}), & \text{if } x > 0 \end{cases}$, $\alpha > 0$

2. Weibull: $\Psi_\alpha(x) = \begin{cases} \exp(-(-x)^\alpha), & \text{if } x \leq 0 \\ 1, & \text{if } x > 0 \end{cases}$, $\alpha > 0$

3. Gumbel: $\Lambda(x) = \exp(-e^{-x})$, $x \in \mathbb{R}$

Finally, we also want to discuss large deviation techniques, which can be e.g. used to study the closeness of $P(c_n^{-1}(M_n - d_n) \leq x)$ and $H(x)$ in (0.3) or which can be employed

(5) to refine results in connection with central limit theorems.

1. Some prerequisites

Denote by R the overnight interest rate at future time $t>0$ or the first arrival time of an insurance claim

$\longrightarrow$ outcome of R uncertain

$\longrightarrow$ model for R:

R random variable (r.v.) on a probability space $(\Omega, \mathcal{F}, P)$

1. Ω:

Def. 2.1 (Sample space)
A set $\Omega \neq \emptyset$ representing the collection of all possible outcomes of a random experiment is called sample space

Ex.: Throw a dic $\longrightarrow \Omega = \{1, 2, \dots, 6\}$

2. $\mathcal{F}$:

Def. 2.2 (σ-algebra on Ω)
A family $\mathcal{F}$ of subsets of Ω is called σ-algebra (on Ω), if

(i) $\emptyset \in \mathcal{F}$

(ii) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$ $(A^c := \Omega \setminus A)$

(iii) $A_1, A_2, \dots \in \mathcal{F} \Rightarrow \bigcup_{i \geq 1} A_i \in \mathcal{F}$

Interpretation: $\mathcal{F}$ is e.g. a model for the entirety of market information up to time t.
The elements $A \in \mathcal{F}$ are called events

Ex: Let S_t be the price of a stock at time t
$\longrightarrow$ $A := "S_t \geq 100\$"$ market event

Ex: $A = \emptyset$ impossible event as e.g. $A = "S_t = \infty" \in \mathcal{F}$
(Def. 2.2(i)) $\longrightarrow$ $A^c = "0 \leq S_t < \infty" \in \mathcal{F}$ (Def. 2.2(ii))

Ex.: Denote by $S_t^{(i)}$, $i \in \mathbb{N}$ the price of the i-th stock at time t

(7) Consider the events

$A_i = "S_t^{(i)} \geq 10\$"$

$\Rightarrow$ $A := "$ at least one stock has a price $\geq 10\$$ at time $t"$

$= \bigcup_{i \in \mathbb{N}} A_i \in \mathcal{F}$ (Def. 2.2(iii))

Ex: $\Omega = \{1, 2, \dots, 6\}$, $\mathcal{F} := \mathcal{P}(\Omega)$ class of all subsets of Ω

$A = \{1, 2, 3\} \in \mathcal{F}$ event that one throws 1, 2 or 3

Ex: (Borel σ-algebra $\mathcal{B}(\mathbb{R})$)

$\Omega = \mathbb{R}$, $\mathcal{F} = \mathcal{B}(\mathbb{R})$, where

$\mathcal{B}(\mathbb{R})$:= smallest σ-algebra, which contains the intervals $[a, b)$, $a \leq b$

= intersection of all σ-algebras $\mathcal{F}_i$ containing the intervals $[a, b)$, $a \leq b$.

3.P:

Def. 2.3 (Probability measure)

A function $P: \mathcal{F} \longrightarrow [0, 1]$

is called probability measure on $(\Omega, \mathcal{F})$, if

(i) $P(\emptyset) = 0$, $P(\Omega) = 1$

(ii) $A_1, \dots, A_n, \dots \in \mathcal{F}$ with $A_i \cap A_j = \emptyset$, $i \neq j$ (disjoint)

$\Rightarrow P(\bigcup_{i \geq 1} A_i) = \sum_{i \geq 1} P(A_i)$ (σ-additivity)

Interpretation: $P(A) =$ " the prob. that the event A occurs "

Useful properties:

(i) $A \subseteq B \Rightarrow P(A) \leq P(B)$

(e.g. $A = "S_t \geq 100\$" \subseteq B = "S_t \geq 90\$"$

(ii) $P(\bigcup_{i \geq 1} A_i) \leq \sum_{i \geq 1} P(A_i)$ for not necessarily disjoint events

(e.g $A_i = "S_t^{(i)} \geq 10\$" \Rightarrow A = \bigcup_{i \geq 1} A_i =$ "

at least one stock $\geq 10\$$ at time $t"$

(iii) $\lim_{i\to\infty} P(A_i) = P(A)$, if $A_1 \subset A_2 \subset \dots$ and $A = \bigcup_{i\geq 1} A_i$

(e.g. $A_i = \{S_t \geq \frac{100}{i}\}$, $i \in \mathbb{N}$)

Examples of prob. meas.

a) $\Omega = \{1,2,\dots,6\}$, $\mathcal{F} = \mathcal{P}(\Omega)$ class of all subsets of Ω

$P(A) := (\text{number of elements in } A)/6$

e.g. $P(\{1,2,3\}) = \frac{1}{2}$ $\longrightarrow$ prob. that one throws 1, 2 or 3

b) Dirac measure δ_ω:

$$\delta_\omega(A) := \begin{cases} 1, & \omega \in A \\ 0, & \omega \notin A \end{cases}$$

c) Lebesgue(-Borel) measure on $[0,1]$

$\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}(\mathbb{R}) \cap [0,1] = \{A \cap [0,1] : A \in \mathcal{B}(\mathbb{R})\}$

One shows that there exists a unique prob. meas.

$\lambda : \mathcal{F} \longrightarrow [0,1]$ s.t.

$\lambda([a,b)) = b-a$ ← length of the interval $[a,b)$

for all $0 \leq a \leq b \leq 1$

Ex.: S uniformly distributed on $[0,1]$

$\iff P(a \leq S \leq b) = b-a = \lambda([a,b))$

$\implies$ distr. of S = Lebesgue meas. on $[0,1]$

Def. 2.4 (Random variable)

A function

$$X : \Omega \longrightarrow \mathbb{R}$$

is called random variable (r.v.) on $(\Omega, \mathcal{F}, P)$, if the set

$$A := \{\omega \in \Omega : X(\omega) < t\}$$

is an event for all $t \in \mathbb{R}$, i.e.

$A \in \mathcal{F}$ (collection of all events)

(9) $\mathcal{F}$ entirely of market information up to time T

$\longrightarrow$ natural to assume that

$\{S_T < t\}$ ← stock price at time T

is a (possible) market event for all t (#), i.e.

$$\{S_T < t\} \in \mathcal{F} \text{ for all } t$$

$\Longrightarrow$ S_T random variable

Def. 2.5 (Random vector)

A function $X : \Omega \longrightarrow \mathbb{R}^n$

with $X(\omega) = (X_1(\omega), \dots, X_n(\omega))$

is called random vector, if $X_1, \dots, X_n$ are r.v.'s

Def. 2.6 (Stochastic process)

Let $\mathcal{I} \neq \emptyset$ be an index set (e.g. $\mathcal{I} = [0,1]$)

A collection

$$\{X_i\}_{i \in \mathcal{I}}$$

of random variables is called stochastic process.

$\mathcal{I}$ is the parameter space.

Ex.: $\mathcal{I} = [0, \infty)$, S_t price of stock at time t

$\longrightarrow$ $\{S_t\}_{0 \le t < \infty}$ stoch. proc.

Def. 2.7 (P-null sets)

A event

$$A \in \mathcal{F}$$

s.t.

$$P(A) = 0$$

is called P-null set

Ex.: $A = \{S_t = \infty\}$ should be a P-null set, i.e. $P(A) = 0$.

↖ stock price

Def. 2.8 (P-almost everywhere properly)

We say a property Π holds P-almost everywhere (P-a.e.), if there ex. a null set $N \in \mathcal{F}$ s.t. Π holds for all $\omega \in N^C = \Omega - N$

Ex.: X_i i-th fire loss claim, $i=1,2$

$\Pi = "X_1 \le X_2"$

$\longrightarrow$ Π holds P-a.e. $\Longleftrightarrow$

$X_1(\omega) \le X_2(\omega)$

for all $\omega \in N^c = \Omega \setminus N$ for a null set $N \in \mathcal{F}$.

Def. 2.9 (Integral w.r.t. P)

Let $X \ge 0$ be a non-negative r.v. Then

$$E[X] \overset{\text{def.}}{=} \int_\Omega X(\omega) P(d\omega) \leftarrow \text{notation}$$

$$\overset{\text{def}}{=} \lim_{n\to\infty} \sum_{i=1}^{n\cdot 2^n} i \cdot 2^{-n} P(A_{i,n}),$$

where

$$A_{i,n} := \begin{cases} \{\omega \in \Omega : (i+1)2^{-n} > X(\omega) \ge i\cdot 2^{-n}\}, & i = 0, \dots, n2^n - 1 \\ \{\omega \in \Omega : X(\omega) \ge n\}, & i = n\cdot 2^n \end{cases}$$

For general X we def.

$$E[X] := \int_\Omega X(\omega) P(d\omega) :=$$

$$E[X^+] - E[X^-] \qquad (x^+ = \max(x,0),\ x^- = \max(-x,0)),$$

provided

$$E[X^+], E[X^-] < \infty \qquad (*)$$

If $(*)$ holds, X is called (P-) integrable.

Ex.: $\Omega = [0,1]$, $P = \lambda$ (uniform distribution)

$X(\omega) = \omega^2$

$$\Rightarrow \quad E[X] = \int_\Omega X(\omega) P(d\omega) = \int_{[0,1]} \omega^2 \lambda(d\omega)$$

$$= \int_0^1 \omega^2 d\omega = \frac{1}{3}\omega^3 \Big|_{\omega=0}^{1} = \frac{1}{3}$$

Properties:

(i) $E[\alpha X + \beta Y] = \alpha E[X] + \beta E[Y]$, $\alpha, \beta \in \mathbb{R}$ (linearity)

(ii) If $X \le Y$ P-a.e. then

$$E[X] \le E[Y]$$

(11) Ex.: $X \in \mathbb{N}_0$ r.v.

$$\Rightarrow \quad X = \sum_{n \geq 0} n \cdot \underbrace{\mathbb{1}_{\{X=n\}}}_{=Y_n \text{ r.v.'s}} = \sum_{n \geq 0} n \cdot Y_n$$

$$\mathbb{1}_A(\omega) := \begin{cases} 1, & \omega \in A \\ 0 & \text{else} \end{cases}$$

Using (i) we obtain that

$$E[X] = E\Big[\sum_{n \geq 0} n \cdot Y_n\Big] = \sum_{n \geq 0} n \cdot \underbrace{E[Y_n]}_{=P(\{X=n\})}$$

$$= \sum_{n \geq 0} n \cdot P(\{X=n\}).$$

Fundamental concept of prob. th. $\longrightarrow$

<u>Conditional expectation</u> $E[X|\mathcal{A}]$ of a r.v. X w.r.t. a σ-algebra $\mathcal{A}$ $(\subseteq \mathcal{F})$

<u>Heuristically</u>:

$$E[X|\mathcal{A}] = \text{“}E[X|A], A \in \mathcal{A}\text{”},$$

where $E[X|A] \overset{\text{def.}}{=} \frac{1}{0 \neq P(A)} E[\mathbb{1}_A \cdot X]$ cond. expectation of X given the event A

<u>Interpretation</u>: $\mathcal{A}$ entirety of market inf. up to time $t < T$ naturel

$\longrightarrow$ $\mathcal{A} \subseteq \mathcal{F}$ market inf. up to time T.

S_T stock price at time T

$\longrightarrow$ $E[S_T|\mathcal{A}]$ expected terminal stock price given the entirety of market inf. up to time $t < T$

<u>Or more mathematically</u>:

<u>Def. 2.10</u> (cond. expectation w.r.t. $\mathcal{A}$) ↙ in general difficult to calculate

Let X be a r.v. s.t. $E[|X|] < \infty$.

Then the <u>expected value of X given $\mathcal{A}$</u> is the

unique r.v. Y s.t.

$$E[1_A \cdot X] = E[1_A \cdot Y]$$

for all $A \in \mathcal{A}$

and

$$\{Y \leq q\} \in \mathcal{A} \text{ for all } q \in \mathbb{R}.$$

The r.v. Y is denoted by $E[X|\mathcal{A}]$

→ Properties of $E[X|\mathcal{A}]$:

(i) $E[\alpha X + \beta Y|\mathcal{A}] = \alpha E[X|\mathcal{A}] + \beta E[Y|\mathcal{A}]$ (linearity)

(ii) $E[E[X|\mathcal{A}]] = E[X]$

(iii) $E[X|\mathcal{A}] = X$, if X r.v. on $(\Omega, \mathcal{A}, P)$

(iv) $E[X|\mathcal{A}] = E[X]$, if X is independent of $\mathcal{A}$, i.e.

$$P(\{X \leq t\} \cap A) = P(X \leq t) \cdot P(A) \text{ for all } t \in \mathbb{R}, A \in \mathcal{A}$$

(v) $E[X|\mathcal{A}] = E[E[X|\mathcal{B}]|\mathcal{A}]$ if $\mathcal{A} \subseteq \mathcal{B}$ σ-algebra

Consider the σ-algebra $\mathcal{F}_t$ as a model for the entirety of market inf. up to time t

→ observation: market inf. $\mathcal{F}_t$ grows over time, i.e.

$$\mathcal{F}_{t_1} \subset \mathcal{F}_{t_2} \subset \mathcal{F}_{t_3} \subset \dots,$$

if $t_1 \leq t_2 \leq t_3 \dots$

→ The family $\mathcal{F}_t$, $0 \leq t \leq T$ is a market information flow

More generally:

Def. 2.11 (Filtration $\mathcal{F}_t$)

Let $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ be a family of σ-algebras on $(\Omega, \mathcal{F}, P)$ s.t.

$$\mathcal{F}_{t_1} \subset \mathcal{F}_{t_2} \quad (\subset \mathcal{F})$$

for all $0 \leq t_1 < t_2 \leq T$.

Then $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ is called filtration on $(\Omega, \mathcal{F}, P)$.

Ex. 2.12: Denote by $\mathcal{F}_t$ the entirety of information based on the stock price dynamics

S_u, $0 \le u \le t$ (efficient market hypothesis; E. Fama, Nobel price 2013)

Or more mathematically

$\widehat{\mathcal{F}}_t$ = smallest σ-alg. containing the events $\{S_u \in A\}$, $A \in \mathcal{B}(\mathbb{R})$, $0 \le u \le t$

$\longrightarrow$ short hand notation: $\widehat{\mathcal{F}}_t = \sigma(S_u ; 0 \le u \le t)$

$\longrightarrow$ fundamental concept:

Def. 2.13 (Martingale)

Let $\{X_t\}_{0 \le t \le T}$ be a stoch. proc. and $\{\widehat{\mathcal{F}}_t\}_{0 \le t \le T}$ a filtration on $(\Omega, \mathcal{F}, P)$.

Assume that X_t is $\widehat{\mathcal{F}}_t$-adapted, i.e.

$$\{X_t < \alpha\} \in \widehat{\mathcal{F}}_t$$

for all $\alpha \in \mathbb{R}$, $0 \le t \le T$

Further, require that

$$E[|X_t|] < \infty$$

for all $0 \le t \le T$

and

$$E[X_t \mid \widehat{\mathcal{F}}_s] = X_s \quad P\text{-a.e.}$$

for all $0 \le s \le t \le T$.

Then $\{X_t\}_{0 \le t \le T}$ is called a martingale w.r.t. $\widehat{\mathcal{F}}_t$

Interpretation (risk neutral world)

$\widehat{\mathcal{F}}_t$ entirety of market inf. up to time t

r market interest rate

$\widetilde{S}_t = e^{-r \cdot t} S_t$ discounted stock price at time t

Assumption: $\widetilde{S}_t$, $0 \le t \le T$ martingale w.r.t. Q, i.e.

$$E_Q[\widetilde{S}_t \mid \widehat{\mathcal{F}}_s] = \widetilde{S}_s \quad Q\text{-a.e.}, \quad 0 \le s \le t \le T$$

$\Longrightarrow$ $E_Q[\cdot]$ on both sides, $s = 0$, $t = T$

$$E_Q[\underbrace{\widetilde{S}_T}_{= e^{-r \cdot T} S_T}] = \widetilde{S}_0 \iff E_Q[S_T] = e^{r \cdot T} S_0$$

$\Longrightarrow$ expected return from S_T (w.r.t. Q) is the return of a deposit in a bank account (or a risk-free bond)

$\longrightarrow$ $(\Omega, \mathcal{F}, Q)$ risk neutral world, Q risk neutral prob. meas. or martingale measure in mathematical finance

Rem.: fair price of an option = $E_Q[$discounted payoff of the option$]$

$\longrightarrow$ important example of a martingale

Def. 2.14 (Brownian motion)

A stoch. proc. $B_t, 0 \le t \le T$ on $(\Omega, \mathcal{F}, P)$ is called Brownian motion (Bm) iff

(i) $B_0 = 0$ P-a.e.

(ii) independent increments:

$B_{t_2} - B_{t_1}, B_{t_2} - B_{t_3}, \dots, B_{t_n} - B_{t_{n-1}}$ independent r.v.'s, if $0 \le t_1 \le t_2 < \dots < t_n$

Recall: $X_1, \dots, X_n$ r.v.'s indep. $\iff$ $P(X_1 \le x_1, \dots, X_n \le x_n) = P(X_1 \le x_1) \cdot \dots \cdot P(X_n \le x_n)$ for all $x_1, \dots, x_n \in \mathbb{R}$

(iii) normal stationary increments:

$B_t - B_s \overset{d}{=} B_{t-s}$, $0 \le s \le t$ and $B_{t-s} \sim N(0, t-s)$

Recall: (i) $X \overset{d}{=} Y \iff P(X \le x) = P(Y \le x)$ for all x ($\leftarrow$ distribution of X)

(ii) $X \sim N(\mu, \sigma^2) \iff$

$$P(X \le x) = \int_{-\infty}^{x} f_X(y)\, dy \quad \text{for}$$

the probability density

$$f_X(y) := \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-(y-\mu)^2/2\sigma^2\right)$$

Ex.: N_t Poisson process with intensity $\lambda > 0$

$\longrightarrow$ $N_t - \lambda t$, $0 \le t \le T$ martingale

2. Risk theory

→ area of insurance mathematics, where typically methods of extreme value theory come into play

Here, we want to study the impact of both small claims and large claims on risk theory models

2.1 Ruin theory

Cramér (1930) introduced a measure for risk in an insurance portfolio called ruin probability

→ probability for ruin in a portfolio

→ Basic notions of ruin theory:

Risk process, Ruin probability, Net Profit condition

Def. 2.1.1 (Renewal model for $S(t)$)
The total claim amount $S(t)$ in the renewal model is described by

$$S(t) = \sum_{i=1}^{N(t)} X_i,$$

where

(i) $X_i, i \geq 1$ are i.i.d non-negative r.v.'s. Here X_i models the i-th claim size (e.g. fire loss).

(ii) $N(t), t \geq 0$ is the renewal (counting) process which counts the number of claims arriving by time t and which is defined as number of

$$N(t) = \#\{i \geq 1 : T_i \leq t\}, \quad t \geq 0,$$

where

$$T_0 := 0, \quad T_n = W_1 + \ldots + W_n$$

for an i.i.d sequence $W_i, i \geq 1$ of (P-a.e.) positive

r.v.'s.

The r.v. T_i is called the <u>i-th arrival time</u> (of claim X_i), whereas $W_i = T_i - T_{i-1}$ is referred to as the <u>i-th inter-arrival time</u>.

and

(iii) $W_i, i \geq 1$ independent of $X_i, i \geq 1$

Recall: $Y_1, \dots, Y_n$ r.v.'s indep. $\iff$
$P(Y_1 \leq x_1, \dots, Y_n \leq x_n) = P(Y_1 \leq x_1) \cdot \dots \cdot P(Y_n \leq x_n)$
for all $x_1, \dots, x_n \in \mathbb{R}$

<u>Rem. 2.1.2</u> (Poisson proc. as an example of a renewal (proc.)

Consider a <u>homogeneous Poisson process</u>, that is a stoch. proc. $N(t), t \geq 0$ on $(\Omega, \mathcal{F}, P)$ s.t.

(i) $N(0) = 0$ P-a.e.

(ii) N has <u>independent increments</u>:

$N(t_1) - N(t_0), \dots, N(t_n) - N(t_{n-1})$

are indep. r.v.'s for all $0 = t_0 < t_1 < \dots < t_n$, $n \in \mathbb{N}$.

(iii) N has <u>stationary Poisson distributed increments</u>:

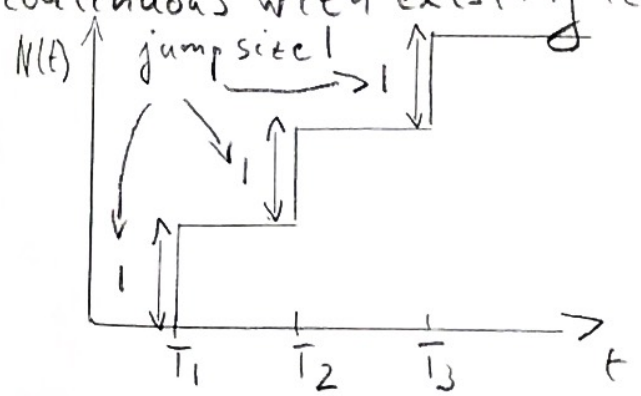
$N(t+h) - N(t) \overset{d}{=} N(h)$ with $N(h) \sim \text{Pois}(\lambda \cdot h)$

for all $t \geq 0$, $h > 0$ for some $\lambda > 0$ (jump intensity)

Recall: $M \sim \text{Pois}(\mu) \iff$
$P(M = k) = e^{-\mu} \frac{\mu^k}{k!}$, $k \in \mathbb{N}_0$.

(iv) With probability 1 the <u>sample paths</u> of $N(t, \omega), t \geq 0$ are <u>cadlag</u>, that is
$\exists\, \tilde{\Omega} \in \mathcal{F}$ with $P(\tilde{\Omega}) = 1$ s.t. for all $\omega \in \tilde{\Omega}$:

$(t \mapsto N(t, \omega))$
right-continuous with existing left limits



(17) One shows that

(i) If N is a hom. Poiss. proc. with intensity λ and arrival times $0 \le T_1 \le T_2 \le \dots$ then N is a renewal proc. with i.i.d. $W_i \sim Exp(\lambda)$, $i \ge 1$, that is

$$P(W_i \le x) = \int_0^x f(x)\, dx \qquad (2.1.1.)$$

with prob density $f(x) = \lambda e^{-\lambda x}$, $x \ge 0$

(ii) Let N be a renewal proc. with i.i.d $W_i \sim Exp(\lambda)$. Then N is a hom. Poisson proc. with intensity $\lambda > 0$.

Ex. 2.1.3 (Renewal model)

$N(t)$, $t \ge 0$ hom. Poisson proc. in Def. 2.1.1. $\longrightarrow$ model for $S(t)$ given by the Cramér-Lundberg-model (0.1)

Def. 2.1.4 (Risk process)

The risk process (or surplus) of the portfolio is defined as

$$U(t) = u + p(t) - S(t), \quad t \ge 0,$$

where

$u > 0$ initial capital (sufficiently large to avoid bankruptcy) ; e.g. reinforced by supervisory authorities

$p(t)$ premium income

Assumption: $p(t) = c \cdot t$, $t \ge 0$,

where $c > 0$ premium rate

